On the Webster Scalar Curvature Problem on the CR Sphere with a Cylindrical-type Symmetry

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Abstract

By variational methods, for a kind of Webster scalar curvature problems on the CR sphere with cylindrically symmetric curvature, we construct some multi-peak solutions as the parameter is sufficiently small under certain assumptions. We also obtain the asymptotic behaviors of the solutions.

Keywords: Webster scalar curvature; variational method; critical point; concentrating solutions.

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1 Introduction and main results

The Webster scalar curvature problem on the CR sphere can be briefly discussed below. Let θ_0 be the standard contact form of the CR manifold $\mathbb{S}^{2n+1}$. Given a smooth function $\bar{\Phi}$ on $\mathbb{S}^{2n+1}$, the Webster scalar curvature problem on $\mathbb{S}^{2n+1}$ consists in finding a contact form θ conformal to θ_0 such that the corresponding Webster

scalar curvature is $\bar{\Phi}$. This problem is equivalent to solve the following equation

$$b_n \Delta_{\theta_0} v(\vartheta) + c_n v(\vartheta) = \bar{\Phi}(\vartheta) v(\vartheta)^{b_n - 1}, \quad \vartheta \in \mathbb{S}^{2n+1}, \tag{1.1}$$

where $b_n = 2 + 2/n$, Δ_{θ_0} is the sub-Laplacian on $(\mathbb{S}^{2n+1}, \theta_0)$ and $c_n = n(n+1)/2$ is the Webster scalar curvature of $(\mathbb{S}^{2n+1}, \theta_0)$. If $v > 0$ solves (1.1), then $(\mathbb{S}^{2n+1}, v^{2/n}\theta_0)$ has the Webster scalar curvature $\bar{\Phi}$. We refer to [14] for a more detailed presentation for this problem.

Using the Heisenberg group $\mathbb{H}^n$ and the CR equivalence $F: \ \mathbb{S}^{2n+1} \setminus \{0, \cdots, -1\} \to \mathbb{H}^n$,

$$F(\vartheta_1, \cdots, \vartheta_{n+1}) = \Big(\frac{\vartheta_1}{1 + \vartheta_{n+1}}, \cdots, \frac{\vartheta_n}{1 + \vartheta_{n+1}}, Re\big(i \frac{1 - \vartheta_{n+1}}{1 + \vartheta_{n+1}} \big) \Big), \tag{1.2}$$

Equation (1.1) becomes, up to an uninfluent constant,

$$- \Delta_{\mathbb{H}^n} u(\zeta) = \Phi(\zeta) u(\zeta)^{\frac{Q+2}{Q-2}}, \ \ \zeta \in \mathbb{H}^n. \tag{1.3}$$

Here $\Delta_{\mathbb{H}^n}$ is the Heisenberg sub-Laplacian, $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$ and Φ corresponds to $\bar{\Phi}$ in the equivalence F.

In this paper, we shall give some existence results for the concentration solutions to problem (1.1) or (1.3), under suitable assumption on the prescribed curvatures. In particular, we shall mainly assume that the prescribed curvature Φ has a natural symmetry, namely a cylindrical-type symmetry.

The Yamabe problem on CR manifolds has been extensively investigated and many interesting results have been obtained, we can refer to [10, 11, 12, 15, 16]. On the contrary, concerning the Webster scalar curvature problem, there are very few results established. In recent years, there has been a growing interest on equations of the same kind of (1.1) or (1.3) and various existence and non-existence results inspired by this topic have been established by several authors, for example, we can refer to [5, 6, 8, 12, 17, 21] and the references therein. However, these results are quite different in nature from the results we shall prove in this paper and do not apply directly to the Webster scalar curvature problem. Recently, in [19], Malchiodi and Uguzzoni obtained an interesting result for problem (1.3) in the perturbative case, i.e., when Φ is assumed to be a small perturbation of a constant.

The aim of this paper is to study a natural case that problem (1.3) has cylindrical curvatures $\Phi(Z,t)=\Phi(|Z|,t)$, which correspond on $\mathbb{S}^{2n+1}$ to curvatures $\bar{\Phi}$ depending only on the last complex variable of $\mathbb{S}^{2n+1}\subset\mathbb{C}^{n+1}$. Concerning this case, in [13] and [7], via the abstract Ambrosetti-Badiale finite dimensional reduction method [1, 2], some results analogous to those found in [3] in the Riemannian setting were verified in the CR setting. However, we should point out that, by Corollary 1.3 below, the solutions found in [13] and [7] are closed to the manifold $\{V_{s,\lambda}(Z,t):\ \lambda>0,\ s\in\mathbb{R}\}$ (for the definition, see (1.5)) and do not have the concentration properties. In the present paper, we will construct some solutions which concentrate on some maximum points of the prescribed curvature as some parameter varies. In particular, our restriction on the prescribed curvature is totally different from that in [13] or [7], more precisely, we only need to impose some kind of flatness condition on each local maximum point of the prescribed curvature.

To state our main results, we first give some notations.

Let us denote a point in $\mathbb{H}^n=\mathbb{C}^n\times\mathbb{R}=\mathbb{R}^{2n}\times\mathbb{R}$ by $\zeta=(Z,t)$, $Z=x+iy$, and by

$$\rho(\zeta)=(|Z|^4+t^2)^{\frac{1}{4}}$$

the homogeneous norm in $\mathbb{H}^n$. In the sequel, we shall always suppose that Φ is continuous and bounded on $\mathbb{H}^n$, and Φ has cylindrical symmetry, i.e. $\Phi(Z,t)=\Phi(|Z|,t)$. Define the space of cylindrically symmetric functions of Folland-Stein Sobolev space $S_0^1(\mathbb{H}^n)$, namely

$$S_{cy}^1(\mathbb{H}^n)=\{u\in S_0^1(\mathbb{H}^n):\ u(Z,t)=u(|Z|,t)\},$$

where $S_0^1(\mathbb{H}^n)$ is defined as the completion of $C_0^\infty(\mathbb{H}^n)$ with respect to the norm

$$\|u\|^2_{S_0^1(\mathbb{H}^n)}=\int_{\mathbb{H}^n}|\nabla_{\mathbb{H}^n}u|^2dZdt.$$

Let us observe that $S_{cy}^1(\mathbb{H}^n)$ is a Hilbert space endowed with the scalar product $\langle u,v\rangle=\int_{\mathbb{H}^n}\nabla_{\mathbb{H}^n}u\cdot\nabla_{\mathbb{H}^n}v\,dV_{\mathbb{H}^n}$. It is known (see [16]) that all the positive cylindrical symmetric solutions in $S_0^1(\mathbb{H}^n)$ to the problem

$$-\Delta_{\mathbb{H}^n}V=V^{\frac{Q+2}{Q-2}},\ \ V\in S_0^1(\mathbb{H}^n) \tag{1.4}$$

are of the form

$$V_{s,\lambda}(Z,t) = c_0 \lambda^{\frac{Q-2}{2}} V_0\Big(\lambda |Z|, \lambda^2 (t-s)\Big), \tag{1.5}$$

where $\lambda > 0$, $s \in \mathbb{R}$, c_0 is a suitable positive constant, and

$$V_0(|Z|,t) = \Big(\frac{1}{(1+|Z|^2)^2 + t^2}\Big)^{\frac{Q-2}{4}}.$$

We first deal with problem (1.3) in the case in which Φ is closed to a constant, namely, the perturbation problem

$$-\Delta_{\mathbb{H}^n} u(Z,t) = (1+\varepsilon K(Z,t)) u(Z,t)^{\frac{Q+2}{Q-2}}, \;\; (Z,t) \in \mathbb{H}^n, \tag{1.6}$$

where $\varepsilon > 0$ is a small parameter, $K(Z,t) = K(|Z|,t)$ is a bounded cylindrical function on $\mathbb{H}^n$ and satisfies that for some $\delta > 0$

$$\begin{aligned} K(Z,t) = \;\; & K(0,\bar{t}) + \xi |Z|^{2\gamma} + a|t-\bar{t}|^{\gamma} + O(((|Z|^2,t)-(0,\bar{t})|^{\gamma+\sigma}), \\ & (Z,t) \in \{(Z,t)\mathbb{H}^n \, : \, \rho((Z,t)-(0,\bar{t})) < \delta\,\}, \end{aligned} \tag{1.7}$$

where ξ, a, γ and σ are some constants depending on $\bar{t}$, $\xi < 0$, $a < 0$, $\gamma \in (1,\, n)$ and $\sigma \in (0,\, 1)$.

On (1.6), we have

Theorem 1.1. *Suppose that $n > 1$ and $K(Z,t)$ satisfies* (1.7) *in the neighborhood of* $(0,\bar{t}^1)$, $(0,\bar{t}^2)$, $(\bar{t}^1 \neq \bar{t}^2)$. *Then there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0,\varepsilon_0)$,* (1.6) *has a solution in $S^1_{cy}(\mathbb{H}^n)$ of the form*

$$u_\varepsilon(Z,t) = \sum_{j=1}^{2} V_{s^j_\varepsilon, \lambda_{\varepsilon,j}}(Z,t) + v_\varepsilon(Z,t)$$

with $\lambda_{\varepsilon,j} \to +\infty$, $s^j_\varepsilon \to \bar{t}^j$ $(j = 1,2)$ and $v_\varepsilon \in S^1_{cy}(\mathbb{H}^n)$, $\|v_\varepsilon\|_{S^1_0(\mathbb{H}^n)} \to 0$ as $\varepsilon \to 0$.

We also consider the non-perturbation problem (1.3). In the following result, we can find some solutions to (1.3) concentrating at exact two points and the distance between these two points can be very large. Moreover, we construct infinitely many solutions for (1.3) or (1.1) under the condition that $\Phi(\zeta)$ has a sequence of strictly local maximum points moving to infinity.

Theorem 1.2. *Assume that $\Phi(Z,t)=\Phi(|Z|,t)$ is bounded and continuous in $\mathbb{H}^n$ $(n\geq 1)$ and satisfies:*

$\Phi(Z,t)$ has a sequence of strictly local maximum point $(0,\bar{t}^j)\in\mathbb{H}^n$ such that $\rho((0,\bar{t}^j))\to+\infty$ and in a small neighbourhood of each $(0,\bar{t}^j)$, there are constants $K_j>0$ and $\gamma_j\in(n,\,n+2)$ such that

$$\Phi(Z,t)=K_j-Q_j((|Z|,t)-(0,\bar{t}^j))+R_j((|Z|,t)-(0,\bar{t}^j)),\ \ \rho((Z,t),(0,\bar{t}^j))<\nu, \tag{1.8}$$

where K_j satisfies $0<c_1\leq K_j\leq c_2<\infty$, $j=1,\cdots$, Q_j satisfies

$$a_0(|Z|^4+t^2)^{\frac{\gamma_j}{2}}\leq Q_j(|Z|,t)\leq a_1(|Z|^4+t^2)^{\frac{\gamma_j}{2}},\ \ j=1,\cdots,$$

for some constants $0<a_0\leq a_1<\infty$ independent of j, and $R_j(|Z|,t)$ satisfies $R_j(|Z|,t)=O((|Z|^4+t^2)^{\frac{\gamma_j+\sigma}{2}})$ for some $\sigma>0$ independent of j.

Then for each small $\mu>0$ and $\bar{t}^{j_1}$, we can find another strictly local maximum point $\bar{t}^{j_2}$, such that (1.3) has a solution in $S_0^1(\mathbb{H}^n)$ of the form

$$u=\sum_{l=1}^{2}[\Phi(0,\bar{t}^{j_l})]^{-n/2}V_{t^{j_l},\lambda_{j_l}}(Z,t)+v(Z,t),$$

where

$$\|v\|_{S_0^1(\mathbb{H}^n)}\leq\mu,\ \ |t^{j_l}-\bar{t}^{j_l}|\leq\mu,\ \ |t^{j_1}-t^{j_2}|\geq\frac{1}{\mu},\ \ \lambda_{j_l}\geq\frac{1}{\mu}.$$

We should point out here that if $\Phi(Z,t)$ is not a constant identically, our assumption that $\Phi(Z,t)$ has at least at two points is necessary for the existence of solutions to our problems. Indeed, if u is a solution to (1.3), then u satisfies the following identity:

$$\int_{\mathbb{H}^n}\langle(Z,2t),\nabla\Phi(Z,t)\rangle u^{\frac{2Q}{Q-2}}dZdt=0,$$

provided the integral is convergent and K is bounded and smooth (see [12]). Hence, $\langle(Z,2t),\nabla\Phi(Z,t)\rangle$ cannot have fixed sign in $\mathbb{H}^n$. As a result, if $\langle(Z,2t),\nabla\Phi(Z,t)\rangle\geq 0$, then (1.7) or (1.8) cannot hold in $\mathbb{H}^n$. If $\langle(Z,2t),\nabla\Phi(Z,t)\rangle\leq 0$, then there are at least two points such that (1.7) or (1.8) is satisfied.

Solutions obtained in Theorem 1.1 above is two-peaked (that is, solutions concentrate at exactly two points simultaneously as $\varepsilon\to 0$). However, a direct corollary from our proof of the theorems is

Corollary 1.3. *Under the assumptions of Theorem 1.1 and Theorem 1.2, problem* (1.3) *does not have single-peaked solution(that is, solutions concentrate at exactly one point) as* $\varepsilon \to 0$.

In fact, if u_ε concentrates exactly at one point, combining the fact $\frac{\partial J_\varepsilon(\eta,\lambda,v)}{\partial \lambda} = 0$ and Lemma 4.1 (where the interaction vanishes), we obtain a contradiction

$$\lambda_\varepsilon^{-\gamma} = o(\lambda_\varepsilon^{-\gamma}).$$

Our techniques consist in the transformation of the problems first into a special form of critical Grushin-type equations and then into a special form of Hardy-Sobolev-type equations on the euclidean space, and the reduction of the problems to a study of a finite-dimensional functional by a type of Lyapunov-Schmidt reduction. In fact, we will give some existence results of concentration solutions on more general Hardy-Sobolev-type equations. We will see later that it is right the transformation of the problems into a Hardy-Sobolev-type equations on the euclidean space that helps us to obtain more precise estimates and furthermore to obtain more precise solutions.

We summarize the rest of the paper. In Section 2, we transform the problems into Hardy-Sobolev-type equations on the euclidean space and give some more general results on the new equations. In Section 3 we give some notations and the sketch of the proof of the main results. The Lyapunov-Schmidt reduction is used to reduce an infinite system to a finite one. Section 4 is devoted to the proof of our main results with degree argument and energy analysis method. For complement, all the basic estimates needed are proved in Section 5.

2 En equivalent problem

In this section, we follow the idea in [7] to derive an equivalent problem of problem (1.3) in the cylindrical case.

Consider the problem (1.3). Recall that the Lie algebra of the left-invariant vector fields on $\mathbb{H}^n$ is generated by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \cdots, n.$$

The sub-elliptic gradient on $\mathbb{H}^n$ is given by $\nabla_{\mathbb{H}^n} = (X_1, \cdots, X_n, Y_1, \cdots, Y_n)$ and the Kohn Laplacian on $\mathbb{H}^n$ is the degenerate-elliptic PDO

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{n} (X_j^2 + Y_j^2).$$

Then by direct computation one can see that

$$X_i^2 u = \frac{\partial^2 u}{\partial x_i^2} + 4y_i \frac{\partial^2 u}{\partial x_i \partial t} + 4y_i^2 \frac{\partial^2 u}{\partial t^2}, \quad Y_i^2 u = \frac{\partial^2 u}{\partial y_i^2} - 4x_i \frac{\partial^2 u}{\partial y_i \partial t} + 4x_i^2 \frac{\partial^2 u}{\partial t^2}.$$

Hence, if $u(Z, t) = u(|Z|, t) > 0$ is cylindrical symmetric (this is natural in the Heisenberg group $\mathbb{H}^n$), then problem (1.3) becomes

$$-\Delta_Z u - 4|Z|^2 u_{tt} = \Phi(|Z|, t) u(|Z|, t)^{\frac{Q+2}{Q-2}}, \quad u > 0, \quad (Z, t) \in \mathbb{R}^{2n} \times \mathbb{R}, \tag{2.1}$$

where Δ_Z is the Eculidean laplacian in $\mathbb{R}^{2n}$.

Equation (2.1) is a special form of the following problem related to the Grushin operator

$$\mathbb{G}u \triangleq -\Delta_y u - 4|y|^2 u_z = \Phi(y, z) u(y, z)^{\frac{Q+2}{Q-2}}, \quad u > 0, \quad (y, z) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}, \tag{2.2}$$

where $Q = m_1 + 2m_2$ is the "appropriate" dimension and $\frac{Q+2}{Q-2}$ is the corresponding critical exponent.

If $\Phi = \Phi(|y|, z)$ and $u = \psi(|y|, z)$ satisfy problem (2.2), then

$$-\psi_{rr}(r, z) - \frac{m_1 - 1}{r} \psi_r(r, z) - 4r^2 \Delta_z \psi(r, z) = \Phi(|y|, z) \psi(r, z)^{\frac{Q+2}{Q-2}}, \tag{2.3}$$

where $r = |y|$.

Define

$$v(r, z) = \psi(\sqrt{r}, z).$$

Then

$$\psi_r(\sqrt{r}, z) = 2\sqrt{r} v_r(r, z), \quad \psi_{rr}(\sqrt{r}, z) = 4r v_{rr}(r, t) + 2v_r(r, z).$$

Hence v satisfies

$$-v_{rr}(r, z) - \frac{m_1}{2r} v_r(r, z) - \Delta_z \psi(r, z) = \frac{\Phi(\sqrt{r}, z)}{4r} v(r, z)^{\frac{Q+2}{Q-2}}, \tag{2.4}$$

that is, $v = v(|y|, z)$ solves the following Hardy-Sobolev-type problem

$$(2.5) \qquad -\Delta u(y,z) = \phi(y,z)\frac{u^{\frac{k+h}{k+h-2}}}{|y|}, \quad (y,z) \in \mathbb{R}^k \times \mathbb{R}^h,$$

where $k = \frac{m_1+2}{2}$, $h = m_2$ and $\phi(y,z) = \phi(|y|, z) = \frac{\Phi(\sqrt{r},z)}{4}$.

As a result, we can summarize the above facts to conclude that

Proposition 2.1. *Let m_1 be even and $\Phi(y,z) = \Phi(|y|, z)$, then $u(y,z) = u(|y|,z)$ solves problem (2.2) if and only if $v(y,z) = u(\sqrt{|y|}, z)$ solves problem (2.5) with $k = \frac{m_1+2}{2}$, $h = m_2$ and $\phi(y,z) = \phi(|y|,z) = \frac{\Phi(\sqrt{r},z)}{4}$. In particular, $u(\zeta) = u(|Z|,t)$ solves problem (1.3) if and only if $v(|Z|,t) = u(\sqrt{|Z|},t)$ solves problem (2.5) with $k = n+1$, $h = 1$. Moreover, there exists $c_n > 0$ such that*

$$(2.6) \qquad \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dZdt = c_n \int_{\mathbb{R}^k \times \mathbb{R}} |\nabla v|^2 dydt.$$

Proof. We only prove (2.6). This can be done by the following calculation:

$$\begin{aligned}
\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dZdt &= \int_{\mathbb{H}^n} \sum_{i=1}^n (|X_i(u)|^2 + |Y_i(u)|^2) dZdt \\
&= \int_{\mathbb{H}^n} \sum_{i=1}^n \Big(\Big|\frac{\partial u}{\partial x_i}\Big|^2 + \Big|\frac{\partial u}{\partial y_i}\Big|^2 + 4(x_i^2 + y_i^2)\Big|\frac{\partial u}{\partial t}\Big|^2 \Big) dZdt \\
&= \omega_{2n} \int_{\mathbb{R}^+ \times \mathbb{R}} \Big(\Big|\frac{\partial u}{\partial r}\Big|^2 + 4r^2 \Big|\frac{\partial u}{\partial t}\Big|^2 \Big) r^{2n-1} drdt \\
&= 2\omega_{2n} \int_{\mathbb{R}^+ \times \mathbb{R}} \Big(\Big|\frac{\partial v}{\partial r}\Big|^2 + \Big|\frac{\partial v}{\partial t}\Big|^2 \Big) r^{k-1} drdt \\
&= \frac{2\omega_{2n}}{\omega_k} \int_{\mathbb{R}^k \times \mathbb{R}} |\nabla v|^2 dydt,
\end{aligned}$$

where (and in the sequel) ω_N is the measure of the $N-1$ dimensional sphere S^{N-1}. $\square$

In the sequel, we will consider a more general problem, that is

$$(2.7) \quad -\Delta u(y,z) = \phi(y,z)\frac{u^{\frac{N}{N-2}}}{|y|}, \quad u > 0, \ (y,z) \in \mathbb{R}^k \times \mathbb{R}^h = \mathbb{R}^N, \ (k \geq 2, \ h \geq 1).$$

Consider the limiting problem

$$(2.8) \qquad -\Delta u = \frac{u^{\frac{N}{N-2}}}{|y|}, \ \ u > 0, \ \ x \triangleq (y,z) \in \mathbb{R}^N, u \in D^{1,2}(\mathbb{R}^N),$$

where

$$D^{1,2}(\mathbb{R}^N) = \{u \in L^{2(N-1)/(N-2)}(|y|, \mathbb{R}^N) : \ |\nabla u| \in L^2(\mathbb{R}^N)\}$$

and $D^{1,2}(\mathbb{R}^N)$ endows the norm $\|u\| \triangleq (\int_{\mathbb{R}^N} |\nabla u|^2 dx)^{1/2}$, which is induced by the inner produce $\langle u, v\rangle = \int_{\mathbb{R}^N} \nabla u \nabla v dx$. It is known from [7] that for $\zeta \in \mathbb{R}^h$, $\lambda > 0$, functions

$$U_{\zeta,\lambda}(x) = \frac{[(N-2)(k-1)]^{\frac{N-2}{2}} \lambda^{\frac{N-2}{2}}}{\Big((1+\lambda|y|)^2 + \lambda^2|z-\zeta|^2\Big)^{\frac{N-2}{2}}}$$

solve (2.8).

Corollary 2.2. *$U_{0,1} = 2^{-n}V_{0,1}$ and $c_0 = (2n)^n$, where c_0 and $V_{0,1}$ are defined by* (1.5).

Proof. By (1.4), (2.6) and (2.8), we can deduce from Proposition 2.1 that $U_{0,1} = 2^{-n}V_{0,1}$. Moreover, by direct calculation, we see $c_0 = (2n)^n$. □

We first consider the case in which $\phi(y, z)$ is a perturbation of a constant, that is $\phi(y, z) = 1 + \varepsilon K(y, z)$. Suppose that for some $\delta > 0$

$$\begin{aligned} K(x) = \ & K(0, \bar\eta) + \sum_{i=1}^{k} \xi_i |y|^\gamma + \sum_{i=1}^{h} a_i |z_i - \bar\eta_i|^\gamma + O(|x-(0,\bar\eta|^{\gamma+\sigma}), \\ & x \in \{x \in \mathbb{R}^N (N>3) \ : \ |x-(0,\bar\eta)| < \delta\}, \end{aligned} \tag{2.9}$$

where ξ_i, a_j, γ and σ are some constants depending on $\bar\eta$, ξ_i, $a_j \neq 0$ for $i = 1, \cdots, k$, $j = 1, \cdots, h$, $\gamma \in (1,\ N-2)$ and $\sigma \in (0,\ 1)$. Set $\xi = (\xi_1, \cdots, \xi_k)$, $\mathbf{a} = (a_1, \cdots, a_h)$. Define

$$g(\pi_1, \pi_2, \gamma, \xi, \mathbf{a}) = \frac{\pi_1}{k} \sum_{j=1}^{k} \xi_j + \frac{\pi_2}{h} \sum_{j=1}^{h} a_j,$$

where

$$\pi_1 = \int_{\mathbb{R}^N} \frac{|y|^\gamma (1-|y|^2-|z|^2) dx}{|y|[(1+|y|)^2+|z|^2]^N}, \quad \pi_2 = \int_{\mathbb{R}^N} \frac{|z|^\gamma (1-|y|^2-|z|^2) dx}{|y|[(1+|y|)^2+|z|^2]^N}.$$

We remark that by Lemma 5.9, $\pi_1 < 0$, $\pi_2 < 0$.

Suppose that

$$g(\pi_1, \pi_2, \gamma, \xi, \mathbf{a}) > 0. \tag{2.10}$$

Define

$$\Lambda := \Big\{(0, \bar\eta) \in \mathbb{R}^N \ \ : \ D_x K(x)\big|_{x=(0,\bar\eta)} = 0, \ \ K(x) \text{ satisfies } \ (2.9) \ \text{ and } \ (2.10)\Big\}.$$

The following result is corresponding to Theorem 1.1.

Theorem 2.3. *Suppose that $K(y,z)$ is bounded and continuous in $\mathbb{R}^N$ $(N > 3)$, $\phi(y,z) = 1 + \varepsilon K(y,z)$, Λ contains at least two points. Then for each $\bar\eta^1$, $\bar\eta^2 \in \Lambda$, $\bar\eta^1 \neq \bar\eta^2$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0,\varepsilon_0)$, (2.7) has a solution of the form*

$$u_\varepsilon = \sum_{j=1}^{2} U_{\eta_\varepsilon^j, \lambda_{\varepsilon,j}} + v_\varepsilon$$

with $\lambda_{\varepsilon,j} \to +\infty$, $\eta_\varepsilon^j \to \bar\eta^j$ $(j = 1,2)$ and $\|v_\varepsilon\| \to 0$ as $\varepsilon \to 0$.

We also construct some solutions to (2.7) which concentrate exactly at two points between which the distance can be very large. This result is a counterpart of Theorem 1.3.

Theorem 2.4. *Assume that ϕ is bounded and continuous in $\mathbb{R}^N$ and satisfies:*

$\phi(y,z)$ has a sequence of strictly local maximum point $(0,\bar\eta^j) \in \mathbb{R}^N$ $(N \geq 3)$ such that $|\bar\eta^j| \to +\infty$ and in a small neighbourhood of each $\bar\eta^j$, there are constants $K_j > 0$ and $\gamma_j \in (N-2,\, N)$ such that

$$\phi(x) = K_j - Q_j(x - (0,\bar\eta^j)) + R_j(x - (0,\bar\eta^j)),\ x = (y,z) \in B_\nu(0,\bar\eta^j), \tag{2.11}$$

where K_j satisfies $0 < c_1 \leq K_j \leq c_2 < \infty$, $j = 1,\cdots$, Q_j satisfies

$$a_0|x|^{\gamma_j} \leq Q_j(x) \leq a_1|x|^{\gamma_j},\ \ j = 1,\cdots,$$

for some constants $0 < a_0 \leq a_1 < \infty$ independent of j, and $R_j(x)$ satisfies $R_j(x) = O(|x|^{\gamma_j+\sigma})$ for some $\sigma > 0$ independent of j.

Then for each small $\mu > 0$ and $\bar\eta^{j_1}$, we can find another strictly local maximum point $\bar\eta^{j_2}$, such that (2.7) has a solution of the form

$$u = \sum_{l=1}^{2} K_{j_l}^{(2-N)/2} U_{\eta^{j_l}, \lambda_{j_l}} + v,$$

where

$$\|v\| \leq \mu,\ \ |\eta^{j_l} - \bar\eta^{j_l}| \leq \mu,\ \ |\eta^{j_1} - \eta^{j_2}| \geq \frac{1}{\mu},\ \ \lambda_{j_l} \geq \frac{1}{\mu}.$$

3 Notations and preliminary results

The functional corresponding to (2.7) can be defined as

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx-\frac{N-2}{2(N-1)}\int_{\mathbb{R}^N}\phi(y,z)\frac{|u|^{\frac{2(N-1)}{N-2}}}{|y|}dx,\ \ u\in D^{1,2}(\mathbb{R}^N).$$

In what follows, we mainly concentrate on the case $\phi(y,z)=1+\varepsilon K(y,z)$. Since the case for non-perturbation in Theorem 2.4 is similar, we will give a sketch to the proof of Theorem 2.4 in Section 4.

We will restrict our arguments to the existence of that particular solution of (2.7) that concentrates, as $\varepsilon\to 0$, at $\bar\eta^1,\bar\eta^2$, that is a solution of the form

$$u_\varepsilon=\sum_{j=1}^{2}U_{\eta_\varepsilon^j,\lambda_{\varepsilon,j}}+v_\varepsilon$$

with $\lambda_{\varepsilon,j}\to+\infty$, $\eta_\varepsilon^j\to\bar\eta^j$ $(j=1,2)$ and $\|v_\varepsilon\|\to 0$ as $\varepsilon\to 0$.

For $\eta=(\eta^1,\eta^2)\in\mathbb{R}^h\times\mathbb{R}^h$, $\lambda=(\lambda_1,\lambda_2)\in\mathbb{R}_+\times\mathbb{R}_+$, denote

$$\begin{aligned}E_{\eta,\lambda}=\Big\{v\in D^{1,2}(\mathbb{R}^N):\Big\langle\frac{\partial U_{\eta^j,\lambda_j}}{\partial\lambda_j},v\Big\rangle=\Big\langle\frac{\partial U_{\eta^j,\lambda_j}}{\partial\eta_i^j},v\Big\rangle=0,\\ for\ \ j=1,2,\ \ i=1,\cdots,h\Big\}.\end{aligned}\tag{3.1}$$

For each $(0,\bar\eta^1)$, $(0,\bar\eta^2)\in\Lambda$, $\bar\eta^1\neq\bar\eta^2$, $\mu>0$, set

$$\begin{aligned}D_\mu\ &=\ \Big\{(\eta,\lambda):\ \eta=(\eta^1,\eta^2)\in\overline{B_\mu(\bar\eta^1)}\times\overline{B_\mu(\bar\eta^2)}\subset\mathbb{R}^h\times\mathbb{R}^h,\\ &\qquad\qquad\lambda=(\lambda_1,\lambda_2)\in\Big(\frac{1}{\mu},+\infty\Big)\times\Big(\frac{1}{\mu},+\infty\Big)\Big\},\\ M_\mu\ &=\ \Big\{(\eta,\lambda,v):(\eta,\lambda)\in D_\mu,\ v\in E_{\eta,\lambda},\ \|v\|<\mu\Big\}.\end{aligned}$$

Let

$$J(\eta,\lambda,v)=I(\sum_{j=1}^{2}U_{\eta^j,\lambda_j}+v).\tag{3.2}$$

Now similar to [4, 20], we have the following lemma.

Lemma 3.1. *For $\mu>0$ small, $u=\sum_{j=1}^{2}U_{\eta^j,\lambda_j}+v$ is a positive critical point of $I(u)$ in $D^{1,2}(\mathbb{R}^N)$ if and only if (η,λ,v) is a critical point of $J(\eta,\lambda,v)$ in M_μ.*

On the other hand, it follows from Lagrange multiplier theorem that $(\eta,\lambda,v)\in M_\mu$ is a critical point of $J(\eta,\lambda,v)$ in the manifold M_μ if and only if there are numbers $B_j\in\mathbb{R}$, $C_{ji}\in\mathbb{R}$ for $i=1,\cdots,h, j=1,2$ such that

$$\frac{\partial J(\eta,\lambda,v)}{\partial v}=\sum_{j=1}^{2}B_j\frac{\partial U_{\eta^j,\lambda_j}}{\partial\lambda_j}+\sum_{j=1}^{2}\sum_{i=1}^{h}C_{ji}\frac{\partial U_{\eta^j,\lambda_j}}{\partial\eta_i^j}, \tag{3.3}$$

$$\frac{\partial J(\eta,\lambda,v)}{\partial\lambda_j}=B_j\Big\langle\frac{\partial^2U_{\eta^j,\lambda_j}}{\partial\lambda_j^2},v\Big\rangle+\sum_{l=1}^{h}C_{jl}\Big\langle\frac{\partial^2U_{\eta^j,\lambda_j}}{\partial\lambda_j\partial\eta_l^j},v\Big\rangle, j=1,2, \tag{3.4}$$

(3.5)
$$\frac{\partial J(\eta,\lambda,v)}{\partial\eta_i^j}=B_j\Big\langle\frac{\partial^2U_{\eta^j,\lambda_j}}{\partial\lambda_j\partial\eta_i^j},v\Big\rangle+\sum_{l=1}^{h}C_{jl}\Big\langle\frac{\partial^2U_{\eta^j,\lambda_j}}{\partial\eta_l^j\partial\eta_i^j},v\Big\rangle,\ j=1,2,\ i=1,\cdots,h.$$

In order to verify Theorem 1.1, following the ideas of [20], we show first that for $(\eta,\lambda)\in D_\mu$ given, there exists $v\in E_{\eta,\lambda}$ and scalars $B_j,C_{ji},i=1,\cdots,h,j=1,2$ such that (3.3) is satisfied. Then as in [9], we employ a degree argument to find suitable $(\eta,\lambda)\in D_\mu$ such that (3.4), (3.5) are satisfied.

Throughout this paper we will let $\varepsilon_{ij}=(\lambda_i\lambda_j|\eta^i-\eta^j|^2)^{(2-N)/2}$ for $i\neq j$ and $C_{N,k}=[(N-2)(k-1)]^{N-1}$.

Proposition 3.2. *For $\bar\eta^1$, $\bar\eta^2\in\Lambda$ and $(\eta,\lambda)\in D_\mu$, there exist $\varepsilon_0>0$, $\mu_0>0$ and a C^1-map which, to any $(\eta,\lambda)\in D_\mu$, $\varepsilon\in(0,\varepsilon_0]$, $\mu\in(0,\mu_0]$, associates $v_\varepsilon(\eta,\lambda):D_\mu\to E_{\eta,\lambda}$ such that $v_\varepsilon(\eta,\lambda)$ satisfies* (3.3) *for some $B_j,C_{ji}(i=1,\cdots,h,j=1,2)$. Furthermore, $v_\varepsilon(\eta,\lambda)$ satisfies the following estimate as $\varepsilon\to0$*

$$\|v_\varepsilon(\eta,\lambda)\|=O\Big(\varepsilon\sum_{j=1}^{2}\Big(|\eta^j-\bar\eta^j|^{\gamma_j}+\frac{1}{\lambda_j^{\gamma_j}}\Big)\Big)+O(\varepsilon_{12}^{\frac12+\tau}),$$

where $\tau>0$ is some constant.

Proof. We expand $J(\eta,\lambda,v)$ in the neighborhood $v=0$. For $v\in E_{\eta,\lambda}$ we obtain

$$J(\eta,\lambda,v)=J(\eta,\lambda,0)+\langle f_\varepsilon,v\rangle+\frac12\langle Q_\varepsilon v,v\rangle+R_\varepsilon(v), \tag{3.6}$$

where $f_\varepsilon\in E_{\eta,\lambda}$ is the linear form over $E_{\eta,\lambda}$ given by

$$\langle f_\varepsilon,v\rangle=\ \langle\sum_{j=1}^{2}U_{\eta^j,\lambda_j},v\rangle-\int_{\mathbb{R}^N}\frac{1+\varepsilon K(x)}{|y|}\Big(\sum_{j=1}^{2}U_{\eta^j,\lambda_j}\Big)^{\frac{N}{N-2}}vdx, \tag{3.7}$$

$\langle Q_\varepsilon v, v\rangle$ is the quadratic form on $E_{\eta,\lambda}$ given by

$$\langle Q_\varepsilon v, v\rangle = \|v\|^2 - \frac{N}{N-2}\int_{\mathbb{R}^N} \frac{1+\varepsilon K(x)}{|y|}\Big(\sum_{j=1}^{2} U_{\eta^j,\lambda_j}\Big)^{\frac{2}{N-2}} v^2 dx, \tag{3.8}$$

and $R_\varepsilon(v)$ is the higher order term satisfying

$$D^{(i)}R_\varepsilon(v) = O(\|v\|^{2+\theta-i}),\ i = 1, 2,$$

where $\theta > 0$ is some constant.

From Proposition 3.3, Q_ε is invertible and $\|Q_\varepsilon^{-1}\| \le C$ for some $C > 0$ independent of η, λ and ε. Now following the arguments in [9, 20] we have

$$\frac{\partial J(\eta,\lambda,v)}{\partial v}\bigg|_{E_{\eta,\lambda}} = f_\varepsilon + Q_\varepsilon v + DR_\varepsilon(v).$$

There exists an equivalence between the existence of v such that (3.3) holds for (η, λ, v) and

$$f_\varepsilon + Q_\varepsilon v + DR_\varepsilon(v) = 0. \tag{3.9}$$

We are now in the position to use the argument in [20] to establish the existence of $v(\eta, \lambda)$ such that (3.3) is satisfied for some numbers $B_j, C_{ji}(i = 1, \cdots, h, j = 1, 2)$. Moreover, there exists a constant $C > 0$ such that

$$\|v\| \le C\|f_\varepsilon\|. \tag{3.10}$$

Now we estimate $\|f_\varepsilon\|$. Note that

$$\begin{aligned}\langle f_\varepsilon, v\rangle = &\int_{\mathbb{R}^N} \frac{1}{|y|}\Big(\sum_{j=1}^{2} U_{\eta^j,\lambda_j}^{\frac{N}{N-2}} - (\sum_{j=1}^{2} U_{\eta^j,\lambda_j})^{\frac{N}{N-2}}\Big) v dx \\ &-\varepsilon\int_{\mathbb{R}^N} \frac{K(x)}{|y|}\Big(\sum_{j=1}^{2} U_{\eta^j,\lambda_j}\Big)^{\frac{N}{N-2}} v dx.\end{aligned}$$

On the other hand, by Hölder inequality and Lemma 5.1 in the appendix,

$$\Big|\int_{\mathbb{R}^N} \frac{1}{|y|}\Big(\sum_{j=1}^{2} U_{\eta^j,\lambda_j}^{\frac{N}{N-2}} - (\sum_{j=1}^{2} U_{\eta^j,\lambda_j})^{\frac{N}{N-2}}\Big) v dx\Big|$$

$$
\begin{aligned}
&= \begin{cases} O\Big(\sum_{i\neq j}\int_{\mathbb{R}^N}\frac{1}{|y|}U_{\eta^i,\lambda_i}^{\frac{N}{2(N-2)}}U_{\eta^j,\lambda_j}^{\frac{N}{2(N-2)}}|v|dx\Big) & (2<\dfrac{2(N-1)}{N-2}<3)\\ O\Big(\sum_{i\neq j}\int_{\mathbb{R}^N}\frac{1}{|y|}U_{\eta^i,\lambda_i}^{\frac{2}{N-2}}U_{\eta^j,\lambda_j}|v|dx\Big) & (\dfrac{2(N-1)}{N-2}\geq 3)\end{cases}\\
&= \begin{cases} O\Big(\sum_{i\neq j}\Big(\int_{\mathbb{R}^N}\frac{1}{|y|}U_{\eta^i,\lambda_i}^{\frac{N-1}{N-2}}U_{\eta^j,\lambda_j}^{\frac{N-1}{N-2}}dx\Big)^{\frac{N}{2(N-1)}}\Big)\|v\| & (2<\dfrac{2(N-1)}{N-2}<3)\\ O\Big(\sum_{i\neq j}\Big(\int_{\mathbb{R}^N}\frac{1}{|y|}U_{\eta^i,\lambda_i}^{\frac{4(N-1)}{N(N-2)}}U_{\eta^j,\lambda_j}^{\frac{2(N-1)}{N}}dx\Big)^{\frac{N}{2(N-1)}}\Big)\|v\| & (\dfrac{2(N-1)}{N-2}\geq 3)\end{cases}\\
&= O(\varepsilon_{12}^{\frac{1}{2}+\tau})\|v\|.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^N}\frac{K(x)}{|y|}\Big(\sum_{j=1}^{2}U_{\eta^j,\lambda_j}\Big)^{\frac{N}{N-2}}vdx\Big|\\
=\;& \Big|\sum_{j=1}^{2}\int_{\mathbb{R}^N}\frac{K(x)}{|y|}U_{\eta^j,\lambda_j}^{\frac{N}{N-2}}vdx\Big|+O(\varepsilon_{12}^{\frac{1}{2}+\tau})\|v\|\\
=\;& O\Big(\sum_{j=1}^{2}\Big(|\eta^j-\bar{\eta}^j|^{\gamma_j}+\frac{1}{\lambda_j^{\gamma_j}}\Big)\Big)\|v\|+O(\varepsilon_{12}^{\frac{1}{2}+\tau})\|v\|.
\end{aligned}
$$

As a result, combining the above three equations, we complete the proof. □

Proposition 3.3. *Let $(\eta,\lambda)\in D_\mu$. Then for $\mu>0$, $\varepsilon>0$ sufficiently small, there exists a $\rho>0$ such that*

$$
\|Q_\varepsilon\omega\|\geq\rho\|\omega\|,\quad\forall\ \omega\in E_{\eta,\lambda}.
$$

Proof. By the boundedness of $K(x)$, it suffices to prove the proposition for the case $\varepsilon=0$. The main idea of the proof is similar to that of Lemma 2.3 in [18].

We argue by contradiction. Suppose that there are $\mu_n\to 0$, $(\eta^n,\lambda_n)\in D_{\mu_n}$ and $\omega_n\in E_{\eta^n,\lambda_n}$ such that

$$
\|Q_0\omega_n\|=o(1)\|\omega_n\|, \tag{3.11}
$$

where $o(1)\to 0$ as $n\to\infty$. In (3.11), we may assume $\|\omega_n\|=1$.

For $j=1,2$, let $\tilde{\omega}_{j,n}(x)=\lambda_{j,n}^{(2-N)/2}\omega_n(\lambda_{j,n}^{-1}x+(0,\eta^{j,n}))$. Then $\tilde{\omega}_{j,n}(x)$ is bounded in $D^{1,2}(\mathbb{R}^N)$. Hence we may assume that there is $\omega_j\in D^{1,2}(\mathbb{R}^N)$ such that as $n\to\infty$,

$$
\tilde{\omega}_{j,n}\rightharpoonup\omega_j,\quad\text{weakly in}\quad D^{1,2}(\mathbb{R}^N). \tag{3.12}
$$

Now we verify that $\omega_j = 0$.

Define

$$\begin{aligned}
\tilde{U}_{j,n} &= \lambda_{j,n}^{(2-N)/2} U_{\eta^{j,n},\lambda_{j,n}}(\lambda_{j,n}^{-1}x + (0,\eta^{j,n})) \\
W_{j,n,i} &= \lambda_{j,n}^{-N/2} \frac{\partial U_{\eta^{j,n},\lambda_{j,n}}(P)}{\partial \eta_i^{j,n}}\Big|_{P=\lambda_{j,n}^{-1}x+(0,\eta^{j,n})}, \quad i = 1,\cdots,h, \\
W_{j,n} &= \lambda_{j,n}^{(4-N)/2} \frac{\partial U_{\eta^{j,n},\lambda_{j,n}}(P)}{\partial \lambda^{j,n}}\Big|_{P=\lambda_{j,n}^{-1}x+(0,\eta^{j,n})}.
\end{aligned}$$

$\omega_n \in E_{\eta^n,\lambda_n}$ implies that

$$\tilde{\omega}_{j,n} \in \tilde{E}_n \triangleq \Big\{\varphi \in D^{1,2}(\mathbb{R}^N): \int_{\mathbb{R}^N} \nabla W_{j,n} \nabla \varphi dx = \int_{\mathbb{R}^N} \nabla W_{j,n,i} \nabla \varphi dx = 0\Big\},$$

$j = 1, 2,\ i = 1,\cdots,h$, and

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla \tilde{\omega}_{j,n} \nabla \varphi dx - \frac{N}{N-2} \int_{\mathbb{R}^N} \frac{1}{|y|} \tilde{U}_{j,n}^{\frac{2}{N-2}} \tilde{\omega}_{j,n} \varphi dx \\
= \ & o(1)\|\varphi\|,\ \forall\ \varphi \in \tilde{E}_n.
\end{aligned} \tag{3.13}$$

We claim that ω_j solves

$$-\Delta \omega_j - \frac{N}{N-2} \frac{U_{0,1}^{\frac{2}{N-2}}}{|y|} \omega_j = 0. \tag{3.14}$$

Indeed, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we can find $c_{j,n}$ and $c_{j,n,i}$ such that

$$\varphi_n = \varphi - \sum_{j=1}^{2} \sum_{i=1}^{h} c_{j,n,i} W_{j,n,i} - \sum_{j=1}^{2} c_{j,n} W_{j,n} \in \tilde{E}_n.$$

Since φ has compact support and the support of $W_{l,n,i}$ and $W_{l,n}$ moves to infinity as $n \to \infty$ for $l \neq j$, it is easy to check that $c_{l,n,i} \to 0$, $c_{l,n} \to 0$ as $n \to \infty$ for any $l \neq j$. Moreover, $c_{j,n,i}$ and $c_{j,n}$ are bounded.

Inserting φ_n into (3.13) and letting $n \to \infty$, we see

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla \omega_j \nabla \varphi dx - \frac{N}{N-2} \int_{\mathbb{R}^N} \frac{U_{0,1}^{\frac{2}{N-2}}}{|y|} \omega_j \varphi dx \\
&-c \int_{\mathbb{R}^N} \Big[\nabla \omega_j \nabla \Big(\frac{\partial U_{0,1}}{\partial \lambda}\Big|_{\lambda=1}\Big) - \frac{N}{N-2} \frac{U_{0,1}^{\frac{2}{N-2}}}{|y|} \omega_j \Big(\frac{\partial U_{0,1}}{\partial \lambda}\Big|_{\lambda=1}\Big)\Big] dx \\
&-\sum_{i=1}^{2} c_i \Big[\int_{\mathbb{R}^N} \nabla \omega_j \nabla \Big(\frac{\partial U_{0,1}}{\partial \eta_i}\Big|_{\eta=0}\Big) dx - \frac{N}{N-2} \frac{U_{0,1}^{\frac{2}{N-2}}}{|y|} \omega_j \Big(\frac{\partial U_{0,1}}{\partial \eta_i}\Big|_{\eta=0}\Big)\Big] dx = 0,
\end{aligned} \tag{3.15}$$

where $c=\lim_{n\to\infty}c_{j,n}$ and $c_i=\lim_{n\to\infty}c_{j,n,i}$. On the other hand,

$$\int_{\mathbb{R}^N}\Big[\nabla\omega_j\nabla\Big(\frac{\partial U_{0,1}}{\partial\lambda}\Big|_{\lambda=1}\Big)-\frac{N}{N-2}\frac{U_{0,1}^{\frac{2}{N-2}}}{|y|}\omega_j\Big(\frac{\partial U_{0,1}}{\partial\lambda}\Big|_{\lambda=1}\Big)\Big]dx=0$$

$$\int_{\mathbb{R}^N}\nabla\omega_j\nabla\Big(\frac{\partial U_{0,1}}{\partial\eta_i}\Big|_{\eta=0}\Big)dx-\frac{N}{N-2}\frac{U_{0,1}^{\frac{2}{N-2}}}{|y|}\omega_j\Big(\frac{\partial U_{0,1}}{\partial\eta_i}\Big|_{\eta=0}\Big)\Big]dx=0.$$

Therefore, we obtain

$$\int_{\mathbb{R}^N}\nabla\omega_j\nabla\varphi dx-\frac{N}{N-2}\int_{\mathbb{R}^N}\frac{U_{0,1}^{\frac{2}{N-2}}}{|y|}\omega_j\varphi dx=0,\quad\forall\ \varphi\in C_0^\infty(\mathbb{R}^N),$$

which implies (3.14).

Proceeding as done to prove (3.15), we see from $\tilde\omega_{j,n}\in\tilde E_n$ that

$$\Big\langle\omega_j,\frac{\partial U_{0,1}}{\partial\lambda}\Big|_{\lambda=1}\Big\rangle=\Big\langle\omega_j,\frac{\partial U_{0,1}}{\partial\eta_i}\Big|_{\eta=0}\Big\rangle=0,\ \ i=1,\cdots,h.$$

But, it is verified in [7] that $U_{0,1}$ is non-degenerate. As a result, we can conclude that $\omega_j=0$.

Now since $\omega_j=0$, we see that

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{1}{|y|}\Big(\sum_{j=1}^2U_{\eta^{j,n},\lambda_{j,n}}\Big)^{\frac{2}{N-2}}\omega_n^2dx\ &\le\ \sum_{j=1}^2\int_{\mathbb{R}^N\setminus B_{R/\lambda_{j,n}}(0,\eta^{j,n})}\frac{1}{|y|}U_{\eta^{j,n},\lambda_{j,n}}^{\frac{2}{N-2}}\omega_n^2dx+o(1)\\ &=\ o_R(1)+o(1),\end{aligned}$$

where $o_R(1)\to0$ as $R\to+\infty$. Hence (3.11) implies that

$$\int_{\mathbb{R}^N}|\nabla\omega_n|^2dx\to0,\ \ (n\to\infty).$$

This is a contradiction to $\|\omega_n\|=1$. □

4 Proof of the main results

To solve system (3.4) and (3.5), we need to estimate each term in them.

Lemma 4.1. *Let $(\eta,\lambda)\in D_\mu$, $v(\eta,\lambda)$ be obtained in Proposition 3.2. For $\mu>0$ and $\varepsilon>0$ small enough, we have for $i=1,2$,*

$$\frac{\partial J(\eta,\lambda,v)}{\partial\lambda_i}\ =\ -C_{N,k}b_1\frac{\varepsilon_{12}}{\lambda_i}-\frac{(N-2)C_{N,k}}{2\lambda_i^{\gamma_i+1}}\Big[\frac{b_2^i}{k}\sum_{j=1}^k\xi_j^i+\frac{b_3^i}{h}\sum_{j=1}^ha_j^i\Big]\varepsilon+O\Big(\frac{\varepsilon\varepsilon_{12}}{\lambda_i}\Big)$$

$$+O\Big(\frac{\varepsilon_{12}^{1+\tau}}{\lambda_i}\Big)+O\Big(\frac{\varepsilon}{\lambda_i}\sum_{j=1}^{2}\Big(\frac{1}{\lambda_j^{\gamma_j+\sigma}}+|\eta^j-\bar{\eta}^j|^{\gamma_j+\sigma}\Big)\Big)+O\Big(\frac{\varepsilon}{\lambda_i^{\gamma_i}}|\eta_i-\bar{\eta}^i|\Big),$$

where, $b_1<0$, $b_2^i<0$, $b_3^i<0$ *are defined in Lemmas 5.2 and 5.3 in the Appendix.*

Proof. By direct computation and Lemmas 5.2-5.4, we have

$$\begin{aligned}
&\frac{\partial J(\eta,\lambda,v)}{\partial\lambda_i}\\
= &\Big\langle\sum_{j=1}^{2}U_{\eta^j,\lambda_j}+v,\frac{\partial U_{\eta^i,\lambda_i}}{\partial\lambda_i}\Big\rangle\\
&-\int_{\mathbb{R}^N}\frac{1+\varepsilon K(x)}{|y|}\Big|\sum_{j=1}^{2}U_{\eta^j,\lambda_j}+v\Big|^{\frac{2}{N-2}}\Big(\sum_{j=1}^{2}U_{\eta^j,\lambda_j}+v\Big)\frac{\partial U_{\eta^i,\lambda_i}}{\partial\lambda_i}dx\\
= &\sum_{j=1}^{2}\Big\langle U_{\eta^j,\lambda_j},\frac{\partial U_{\eta^i,\lambda_i}}{\partial\lambda_i}\Big\rangle-\int_{\mathbb{R}^N}\frac{1+\varepsilon h(x)}{|y|}\Big(\sum_{j=1}^{2}U_{\eta^j,\lambda_j}\Big)^{\frac{N}{N-2}}\frac{\partial U_{\eta^i,\lambda_i}}{\partial\lambda_i}dx\\
&-\frac{N}{N-2}\int_{\mathbb{R}^N}\frac{1+\varepsilon h(x}{|y|}\Big(\sum_{j=1}^{2}U_{\eta^j,\lambda_j}\Big)^{\frac{2}{N-2}}\frac{\partial U_{\eta^i,\lambda_i}}{\partial\lambda_i}vdx+O\Big(\frac{1}{\lambda_i}\Big)\|v\|^2\\
= &\int_{\mathbb{R}^N}\frac{1}{|y|}\Big(\sum_{j=1}^{2}U_{\eta^j,\lambda_j}^{\frac{N}{N-2}}-\Big(\sum_{j=1}^{2}U_{\eta^j,\lambda_j}\Big)^{\frac{N}{N-2}}\Big)\frac{\partial U_{\eta^i,\lambda_i}}{\partial\lambda_i}dx\\
&-\varepsilon\int_{\mathbb{R}^N}\frac{K(x)}{|y|}\Big(\sum_{j=1}^{2}U_{\eta^j,\lambda_j}\Big)^{\frac{2}{N-2}}\frac{\partial U_{\eta^i,\lambda_i}}{\partial\lambda_i}dx\\
&-\frac{N}{N-2}\int_{\mathbb{R}^N}\frac{1+\varepsilon K(x)}{|y|}\Big(\sum_{j=1}^{2}U_{\eta^j,\lambda_j}\Big)^{\frac{2}{N-2}}\frac{\partial U_{\eta^i,\lambda_i}}{\partial\lambda_i}vdx+O\Big(\frac{1}{\lambda_i}\Big)\|v\|^2\\
= &-\frac{N}{N-2}\int_{\mathbb{R}^N}\frac{1}{|y|}U_{\eta^i,\lambda_i}^{\frac{2}{N-2}}\frac{\partial U_{\eta^i,\lambda_i}}{\partial\lambda_i}U_{\eta^j,\lambda_j}dx-\varepsilon\int_{\mathbb{R}^N}\frac{K(x)}{|y|}U_{\eta^i,\lambda_i}^{\frac{N}{N-2}}\frac{\partial U_{\eta^i,\lambda_i}}{\partial\lambda_i}dx\\
&+O\Big(\frac{\varepsilon\varepsilon_{12}}{\lambda_i}\Big)+O\Big(\frac{\varepsilon_{12}^{1+\tau}}{\lambda_i}\Big)+O\Big(\frac{\varepsilon}{\lambda_i}\sum_{j=1}^{2}\Big(\frac{1}{\lambda_j^{\gamma_j+\sigma}}+|\eta^j-\bar{\eta}^j|^{\gamma_j+\sigma}\Big)\Big)\\
= &-C_{N,k}b_1\frac{\varepsilon_{12}}{\lambda_i}-\frac{(N-2)C_{N,k}}{2\lambda_i^{\gamma_i+1}}\Big[\frac{b_2^i}{k}\sum_{j=1}^{k}\xi_j^i+\frac{b_3^i}{h}\sum_{j=1}^{h}a_j^i\Big]\\
&+O\Big(\frac{\varepsilon\varepsilon_{12}}{\lambda_i}\Big)+O\Big(\frac{\varepsilon_{12}^{1+\tau}}{\lambda_i}\Big)+O\Big(\frac{\varepsilon}{\lambda_i}\sum_{j=1}^{2}\Big(\frac{1}{\lambda_j^{\gamma_j+\sigma}}+|\eta^j-\bar{\eta}^j|^{\gamma_j+\sigma}\Big)\Big)+O\Big(\frac{\varepsilon}{\lambda_i^{\gamma_i}}|\eta_i-\bar{\eta}^i|\Big).
\end{aligned}$$

□

Similar to the proof of Lemma 4.1, using the estimates in Lemmas 5.5-5.7 in the Appendix we obtain

Lemma 4.2. *Under the same assumption as in Lemma 4.1, we have for* $j, l = 1, 2,\ j \neq l,\ i = 1, \cdots, h$

$$\begin{aligned}\frac{\partial J(\eta,\lambda,v)}{\partial \eta_i^j} &= -\frac{b_4^j a_i^j}{\lambda_j^{\gamma_j-2}}\varepsilon(\eta_i^j-\bar{\eta}_i^j) - C(\eta_i^j-\eta_i^l)\varepsilon_{12} + O\Big(\varepsilon\frac{\lambda_j^2|\eta^j-\bar{\eta}^j|^2}{\lambda_j^{\gamma_j-1}}\Big) \\ &\quad + O\Big(\varepsilon\sum_{l=1}^{2}\Big(\frac{1}{\lambda_l^{\gamma_l-1+\sigma}} + \lambda_l|\eta^l-\bar{z}^l|^{\gamma_l+\sigma}\Big)\Big) + O(\varepsilon\lambda_j\varepsilon_{12}) + O(\lambda_j\varepsilon_{12}^{1+\tau}),\end{aligned}$$

where C is a positive constant and b_4^j is defined in Lemma 5.6.

Lemma 4.3. *For $(\eta,\lambda) \in D_\mu$ and $v(\eta,\lambda)$ obtained in Proposition 3.2, $j = 1, 2$, $i = 1, \cdots, h$,*

$$\begin{aligned}B_j &= O\Big(\lambda_j^2\Big(\sum_{l=1}^{2}\Big(\frac{\varepsilon}{\lambda_l^{\gamma_l+1}} + \varepsilon|\eta^l-\bar{\eta}^l|^{\gamma_l+1}\Big)\Big)\Big) + O(\lambda_j\varepsilon_{12}), \\ C_{ji} &= O\Big(\frac{1}{\lambda_j^2}\Big(\frac{\varepsilon}{\lambda_j^{\gamma_j-2}}|\eta^j-\bar{\eta}^j| + \varepsilon_{12}\Big)\Big) + O\Big(\frac{\varepsilon}{\lambda_j}\sum_{l=1}^{2}\Big(\frac{1}{\lambda_l^{\gamma_l+\sigma}} + |\eta^l-\bar{\eta}^l|^{\gamma_l+\sigma}\Big)\Big).\end{aligned}$$

Proof. For each $\varphi \in D^{1,2}(\mathbb{R}^N)$, there holds

$$\Big\langle \frac{\partial J(\eta,\lambda,v)}{\partial v}, \varphi\Big\rangle = \sum_{j=1}^{2} B_j \Big\langle \frac{\partial U_{\eta^j,\lambda_j}}{\partial \lambda_j}, \varphi\Big\rangle + \sum_{j=1}^{2}\sum_{i=1}^{h} C_{ji}\Big\langle \frac{\partial U_{\eta^j,\lambda_j}}{\partial \eta_i^j}, \varphi\Big\rangle.$$

We take $\varphi = \frac{\partial U_{\eta^j,\lambda_j}}{\partial \lambda_j},\ \frac{\partial U_{\eta^j,\lambda_j}}{\partial \eta_i^j},\ j = 1, 2,\ i = 1, \cdots, h$ into the above equation, and use the fact that

$$\Big\langle \frac{\partial J}{\partial v}, \frac{\partial U_{\eta^j,\lambda_j}}{\partial \lambda_j}\Big\rangle = \frac{\partial J}{\partial \lambda_j}, \qquad \Big\langle \frac{\partial J}{\partial v}, \frac{\partial U_{\eta^j,\lambda_j}}{\partial \eta_i^j}\Big\rangle = \frac{\partial J}{\partial \eta_i^j},$$

then we get a quasi-diagonal linear system with B_j, C_{ji} as unknowns. Using the estimates in Lemmas 4.1, 4.2 and 5.8, the required estimates can be obtained. □

Proof of Theorem 2.3. Let

$$L_\varepsilon = \varepsilon^{-\frac{\gamma_1\gamma_2}{\frac{N-2}{2}(\gamma_1+\gamma_2)-\gamma_1\gamma_2}}.$$

Now to complete the proof, we only need to show that (3.4), (3.5) are satisfied by some $(\eta, \lambda) \in D_\mu$. We will show that for some suitable $\delta > 0$, $m_1 > 0$ small and m_2 large, there exists $(\eta, \lambda) \in D_\mu$ such that

$$(\lambda_1, \lambda_2) \in [m_1 L_\varepsilon^{\gamma_1^{-1}}, m_2 L_\varepsilon^{\gamma_1^{-1}}] \times [m_1 L_\varepsilon^{\gamma_2^{-1}}, m_2 L_\varepsilon^{\gamma_2^{-1}}], \quad \eta = (\eta^1, \eta^2) \in B_{\frac{\delta}{\lambda_1}}(\bar{\eta}^1) \times B_{\frac{\delta}{\lambda_2}}(\bar{\eta}^2)$$

$(\eta, \lambda, v(\eta, \lambda))$ satisfies (3.4) and (3.5).

Employing Lemmas 4.1-4.3, Lemma 5.8, we get the following equivalent form of (3.4), (3.5)

$$\begin{aligned}
\frac{\varepsilon}{\lambda_j^{\gamma_j}}(\eta_i^j - \bar{z}_i^j) &= O\Big(\varepsilon \sum_{l=1}^{2} \Big(\frac{1}{\lambda_l^{\gamma_l+\sigma}} + |\eta^l - \bar{\eta}^l|^{\gamma_l+\sigma}\Big)\Big) \\
&\quad + O(\frac{\varepsilon_{12}}{\lambda_j}), \ \ j = 1, 2, i = 1, \cdots, h, \\
\frac{b_1}{(\lambda_1\lambda_2)^{\frac{N-2}{2}}} + \frac{N-2}{2}\Big[\frac{b_2^j}{k}\sum_{l=1}^{k} \xi_l^j + \frac{b_3^j}{h}\sum_{l=1}^{h} a_l^j\Big]\frac{\varepsilon}{\lambda_i^{\gamma_i}} &= O\Big(\varepsilon \sum_{l=1}^{2} \Big(\frac{1}{\lambda_l^{\gamma_l+\sigma}} + |\eta^l - \bar{\eta}^l|^{\gamma_l+\sigma}\Big)\Big) \\
&\quad + O(\varepsilon\varepsilon_{12} + \varepsilon_{12}^{1+\tau}), \ \ j = 1, 2.
\end{aligned}$$

Let

$$\lambda_1 = t_1 L_\varepsilon^{\gamma_1^{-1}}, \ \ \lambda_2 = t_2 L_\varepsilon^{\gamma_2^{-1}}, \ \ t_j \in [m_1, m_2],$$
$$\eta^1 - \bar{\eta}^1 = \lambda_1^{-1} x^1, \ \ \eta^2 - \bar{\eta}^2 = \lambda_2^{-1} x^2, \ \ (x^1, x^2) \in B_\delta(0) \times B_\delta(0) \subset \mathbb{R}^h \times \mathbb{R}^h.$$

Then since $b_1 < 0$ and $\frac{b_2^j}{k}\sum_{l=1}^{k} \xi_l^j + \frac{b_3^j}{h}\sum_{l=1}^{h} a_l^j > 0$ by (2.10), the above system can be rewritten in the following equivalent one

$$\begin{cases} x^j = o_\varepsilon(1), \ \ j = 1, 2, \\ t_j^{-\gamma_j} - \frac{c_j}{(t_1 t_2)^{\frac{N-2}{2}}} = o_\varepsilon(1), \ \ j = 1, 2, \end{cases}$$

where $c_j = -b_1\Big(\frac{N-2}{2}\Big[\frac{b_2^j}{k}\sum_{l=1}^{k} \xi_l^j + \frac{b_3^j}{h}\sum_{l=1}^{h} a_l^j\Big]\Big)^{-1} > 0$, $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$.

Let

$$\begin{aligned}
f(x^1, x^2) &= (x^1, x^2), \quad (x^1, x^2) \in \Omega_1 := B_\delta(0) \times B_\delta(0), \\
g(t_1, t_2) &= \Big(t_1^{-\gamma_1} - \frac{c_1}{(t_1 t_2)^{\frac{N-2}{2}}}, t_2^{-\gamma_2} - \frac{c_2}{(t_1 t_2)^{\frac{N-2}{2}}}\Big), \\
&\quad (t_1, \, t_2) \in \Omega_2 := [m_1, \, m_2] \times [m_1, \, m_2].
\end{aligned}$$

With the same arguments as in [9], we can prove

$$deg\big((f,g),\,\Omega_1\times\Omega_2,\,0\big)=-1.$$

As a consequence, we complete the proof. □

Proof of Theorem 2.4. The main idea is from [22]. Since all the computations are similar to those in the proof of Theorem 2.2, we only give the sketch.

For simplicity, we assume that $(0,\bar\eta^{j_1})=(0,\bar\eta^1)$ and $(0,\bar\eta^2)$ is another local maximum point of ϕ with $s\triangleq|\bar\eta^1-\bar\eta^2|$ large enough. Define

$$L_1=s^{\frac{(N-2)\gamma_2}{\gamma_1\gamma_2-(\gamma_1+\gamma_2)(N-2)/2}},\;\; L_2=s^{\frac{(N-2)\gamma_1}{\gamma_1\gamma_2-(\gamma_1+\gamma_2)(N-2)/2}}.$$

Proceeding as done in the proof of Proposition 3.2, we do the finite dimensional reduction to obtain $v(\eta,\lambda)$ and the same estimate on $v(\eta,\lambda)$.

We now study the problem

$$\inf\{J(\eta,\lambda,v(\eta,\lambda)):\;\;(\eta,\lambda)\in D_{\mu,2}\},\tag{4.1}$$

where

$$D_{\mu,2}\triangleq\{(\eta,\lambda):\;(\eta,\lambda)\in D_\mu,\;\lambda_j\in[\beta_1L_j,\,\beta_2L_j],\;j=1,2\},$$

$\beta_1>0$ is a small constant and $\beta_2>0$ is a large constant and both of them will be determined later. Problem (4.1) has a minimizer $(\tilde\eta,\tilde\lambda)\in D_{\mu,2}$. In the sequel, We will prove that for s large enough, $(\tilde\eta,\tilde\lambda)$ is an interior point of $D_{\mu,2}$ and thus is a critical point of $J(\eta,\lambda,v(\eta,\lambda))$.

By Lemma 5.1, calculating as in the proof of Proposition 3.2, we obtain

$$\begin{aligned}J(\eta,\lambda,v(\eta,\lambda)) &= J(\eta,\lambda,0)+O(\|v\|^2))\\ &= \sum_{j=i}^{2}I\Big(\frac{U_{\eta^j,\lambda_j}}{\phi(0,\eta^j)^{(N-2)/2}}\Big)-\frac{D\varepsilon_{12}}{\phi(0,\eta^1)^{(N-2)/2}\phi(0,\eta^2)^{(N-2)/2}}\\ &\quad+O\Big(\sum_{j=1}^{2}\Big(|\eta^j-\bar\eta^j|^{2\gamma_j}+\frac{1}{\lambda_j^{2\gamma_j}}\Big)\Big)+O(\varepsilon_{12}^{1+2\tau})\\ &= \sum_{j=i}^{2}\Big[\Big(\frac12\frac{1}{\phi(0,\eta^j)^{N-2}}-\frac{N-2}{2(N-1)}\frac{\phi(0,\bar\eta^j)}{\phi(0,\eta^j)^{N-1}}\Big)A\end{aligned}$$

$$
\begin{aligned}
&+\frac{N-2}{2(N-1)}\frac{1}{\phi(0,\eta^j)^{N-1}}\int_{\mathbb{R}^N}\frac{1}{|y|}Q_j\Big(\frac{x}{\lambda_j}+(0,\eta^j)-(0,\bar\eta^j)\Big)U_{0,1}^{\frac{2(N-1)}{N-2}}\Big]\\
&-\frac{D\varepsilon_{12}}{\phi(0,\eta^1)^{(N-2)/2}\phi(0,\eta^2)^{(N-2)/2}}\\
&+O\Big(\sum_{j=1}^{2}\Big(|\eta^j-\bar\eta^j|^{\gamma_j+\sigma}+\frac{1}{\lambda_j^{\gamma_j+\sigma}}\Big)\Big),
\end{aligned}
$$

where $A=\int_{\mathbb{R}^N}\frac{1}{|y|}U_{0,1}^{\frac{2(N-1)}{N-2}}$ and $D>0$ is a constant independent of s.

We first prove that $|\tilde\eta^j-\bar\eta^j|<\frac{C}{\lambda_j}$.

Notice that

$$
\begin{aligned}
\frac{1}{2}\frac{1}{\phi(0,\eta^j)^{N-2}}-\frac{N-2}{2(N-1)}\frac{\phi(0,\bar\eta^j)}{\phi(0,\eta^j)^{N-1}} &= \frac{1}{2(N-1)}\frac{1}{\phi(0,\bar\eta^j)^{N-2}}\\
&\quad +O(|\eta^j-\bar\eta^j|^{2\gamma_j}),\\
Q_j\Big(\frac{x}{\lambda_j}+(0,\eta)-(0,\bar\eta^j)\Big) &\geq a_0|\eta^j-\bar\eta^j|^{\gamma_j}-C\frac{|x|^{\gamma_j}}{\lambda_j}.
\end{aligned}
$$

Hence from

$$J(\tilde\eta,\tilde\lambda,v(\tilde\eta,\tilde\lambda))\leq J(\bar\eta,\tilde\lambda,v(\bar\eta,\tilde\lambda)),$$

we conclude

$$\sum_{j=i}^{2}a_0|\tilde\eta^j-\bar\eta^j|^{\gamma_j}\leq O\Big(\sum_{j=1}^{2}\Big(|\tilde\eta^j-\bar\eta^j|^{2\gamma_j}+\frac{1}{\lambda_j^{\gamma_j}}\Big)\Big)+O(\varepsilon_{12}^1),$$

which implies $|\tilde\eta^j-\bar\eta^j|<\frac{C}{\lambda_j}$.

At last, we verify that $\tilde\lambda_j\in(\beta_1L_j,\beta_2L_j),\ j=1,2$.

Denote $\tilde\lambda_j=t_jL_j$, $j=1,2$. By the fact $\gamma_j>N-2$, we deduce that there exists $(t_{0,1},t_{0,2})\in\mathbb{R}^+\times\mathbb{R}^+$, such that

$$\sum_{j=1}^{2}\frac{c}{t_{0,j}^{\gamma_j}}-\frac{D}{[t_{0,1}t_{0,2}\phi(0,\bar\eta^1)\phi(0,\bar\eta^2)]^{(N-2)/2}}<-c'<0. \tag{4.2}$$

Let $\lambda_0=(t_{0,1}L_1,\,t_{0,2}L_2)$. Then

$$J(\bar\eta,\lambda_0,v(\bar\eta,\lambda_0))\leq\sum_{j=1}^{2}\frac{1}{2(N-1)}\frac{1}{\phi(0,\bar\eta^j)^{N-2}}A-c_0's^{-\frac{2\gamma_1\gamma_2(N-2)}{2\gamma_1\gamma_2-(\gamma_1+\gamma_2)(N-2)}}, \tag{4.3}$$

for some constant $c_0'>0$. On the other hand, direct computation gives

$$J(\tilde\eta,\tilde\lambda,v(\tilde\eta,\tilde\lambda))\ \geq\ \sum_{j=1}^{2}\frac{1}{2(N-1)}\frac{A}{\phi(0,\bar\eta^j)^{N-2}}$$

$$+c''\sum_{j=1}^{2}\frac{1}{\tilde\lambda_j^{\gamma_j}}-\frac{D\varepsilon_{12}}{\phi(0,\tilde\eta^1)^{(N-2)/2}\phi(0,\tilde\eta^2)^{(N-2)/2}}$$
$$+O\Big(\sum_{j=1}^{2}\frac{1}{\tilde\lambda_j^{\gamma_j+\sigma}}+\varepsilon_{12}^{1+2\tau}\Big)$$

Hence, employing $J(\tilde\eta,\tilde\lambda,v(\tilde\eta,\tilde\lambda))\le J(\bar\eta,\lambda_0,v(\bar\eta,\lambda_0))$, we see

$$c''\sum_{j=1}^{2}\frac{1}{\tilde\lambda_j^{\gamma_j}}-\frac{D\varepsilon_{12}}{\phi(0,\tilde\eta^1)^{(N-2)/2}\phi(0,\tilde\eta^2)^{(N-2)/2}}\le -c_0's^{-\frac{2\gamma_1\gamma_2(N-2)}{2\gamma_1\gamma_2-(\gamma_1+\gamma_2)(N-2)}}. \tag{4.4}$$

If $\tilde\lambda_1=\beta_1L_1$, then

$$\varepsilon_{12}=\frac{1+o(1)}{(\tilde\lambda_1\tilde\lambda_2|\tilde\eta^1-\tilde\eta^2|^2)^{(N-2)/2}}\le\frac{2}{\beta_1^{N-2}}s^{-\frac{2\gamma_1\gamma_2(N-2)}{2\gamma_1\gamma_2-(\gamma_1+\gamma_2)(N-2)}}.$$

Hence (4.4) implies

$$\frac{c''}{\beta_1^{\gamma_1}}-\frac{2}{\beta_1^{N-2}}\le -c_0'.$$

This is impossible for β_1 small since $\gamma_1>N-2$.

If $\tilde\lambda_1=\beta_2L_1$, then

$$\varepsilon_{12}=\frac{1+o(1)}{(\tilde\lambda_1\tilde\lambda_2|\tilde\eta^1-\tilde\eta^2|^2)^{(N-2)/2}}\le\frac{\bar c}{\beta_1^{(N-2)/2}\beta_2^{(N-2)/2}}s^{-\frac{2\gamma_1\gamma_2(N-2)}{2\gamma_1\gamma_2-(\gamma_1+\gamma_2)(N-2)}}.$$

Hence (4.4) implies

$$-\frac{\bar c}{\beta_1^{(N-2)/2}\beta_2^{(N-2)/2}}\le -c_0'.$$

This is impossible for β_2 (depending on β_1) sufficiently large. Since the same argument can be applied to $\tilde\lambda_2$, we see $\tilde\lambda_j\in(\beta_1L_j,\beta_2L_j),\ j=1,2.$ □

Now we give the proof of Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1 and Theorem 1.2: Define

$$D_{cy}^{1,2}(\mathbb{R}^N)\triangleq\{u\in D^{1,2}(\mathbb{R}^N):\ \ u(y,z)=u(|y|,z)\},$$

then it is well known that the positive critical points of $I|_{D_{cy}^{1,2}(\mathbb{R}^N)}$ are indeed critical points of I in $D^{1,2}(\mathbb{R}^N)$. Hence, since Proposition 2.1 and Corollary 2.2, to complete the proof, it suffices to prove that if $\phi(y,z)$ is cylindrically symmetric in y, then so is v_ε in Proposition 3.2.

Indeed, if $\phi(y,z)=\phi(|y|,z)$, since $\frac{\partial U_{\eta^j,\lambda_j}}{\partial\lambda_j}$, $\frac{\partial U_{\eta^j,\lambda_j}}{\partial\eta^i_j}$ are cylindrically symmetric in y, we can do in the proof of Proposition 3.2 the finite dimensional reduction of J to get the same results in $D^{1,2}_{cy}(\mathbb{R}^N)$. □

5 Appendix

In this section we give some basic estimates used in the previous sections.

Lemma 5.1. *Let $\alpha\ge\beta>1$ such that $\alpha+\beta=\frac{2(N-1)}{N-2}$. Then*

$$\int_{\mathbb{R}^N}\frac{1}{|y|}U^{\frac{N}{N-2}}_{\eta^i,\lambda_i}U_{\eta^j,\lambda_j}dx=C_{N,k}\varepsilon_{ij}\int_{\mathbb{R}^N}\frac{dx}{|y|[(1+|y|)^2+|z|^2]^{\frac{N}{2}}}+(\varepsilon_{ij}^{1+\tau}),$$
$$\int_{\mathbb{R}^N}\frac{1}{|y|}U^{\alpha}_{\eta^i,\lambda_i}U^{\beta}_{\eta^j,\lambda_j}dx=O(\varepsilon_{ij}^{1+\tau}),$$

where τ is some small positive constant.

Proof. We only prove the first estimate for the case $\lambda_1\ge\lambda_2$, all the rest can be proved similarly. Split

$$\begin{aligned}\mathbb{R}^N &= \{\bar{x}:|x|\le\sqrt{\lambda_1\lambda_2}/10\}\\ &\quad\bigcup\{|x|>\sqrt{\lambda_1\lambda_2}/10:|x-\lambda_1(0,\eta^2-\eta^1)|\ge\lambda_1|\eta^2-\eta^1|/10\}\\ &\quad\bigcup\{|x|>\sqrt{\lambda_1\lambda_2}/10:|x-\lambda_1(0,\eta^2-\eta^1)|<\lambda_1|\eta^2-\eta^1|/10\}\\ &=:\Omega_1\cup\Omega_2\cup\Omega_3.\end{aligned}$$

$$\begin{aligned}&\int_{\mathbb{R}^N}\frac{1}{|y|}U^{\frac{N}{N-2}}_{\eta^1,\lambda_1}U_{\eta^2,\lambda_2}dx\\ =&\int_{\mathbb{R}^N}\frac{C_{N,k}}{|y|[(1+|y|)^2+|z|^2]^{\frac{N}{2}}}\frac{dx}{\Big(\frac{\lambda_1}{\lambda_2}(1+\frac{\lambda_2}{\lambda_1}|y|)^2+|\sqrt{\frac{\lambda_2}{\lambda_1}}z-\sqrt{\lambda_1\lambda_2}(\eta^2-\eta^1)|^2\Big)^{\frac{N-2}{2}}}\\ =&\ C_{N,k}\varepsilon_{12}\int_{\Omega_1}\frac{dx}{|y|[(1+|y|)^2+|z|^2]^{\frac{N}{2}}}\Big(1-\frac{(N-2)|y|}{\lambda_1/\lambda_2+\lambda_1\lambda_2|\eta^2-\eta^1|^2}\\ &+\frac{\lambda_2/\lambda_1O(|x|^2)}{\lambda_1/\lambda_2+\lambda_1\lambda_2|\eta^2-\eta^1|^2}\Big)+O(\varepsilon_{12})\int_{\Omega_2}\frac{C_{N,k}dx}{|y|[(1+|y|)^2+|z|^2]^{\frac{N}{2}}}\\ &+O\Big(\frac{1}{\lambda_1^N|\eta^2-\eta^1|^N}\Big)\int_{\Omega_3}\frac{dx}{|y|\Big(\Big(\frac{\lambda_1}{\lambda_2}(1+\frac{\lambda_2}{\lambda_1}|y|)^2+|\sqrt{\frac{\lambda_2}{\lambda_1}}z-\sqrt{\lambda_1\lambda_2}(\eta^2-\eta^1)|^2\Big)^{\frac{N-2}{2}}}\end{aligned}$$

$$
\begin{aligned}
&= C_{N,k}\varepsilon_{12}\int_{\mathbb{R}^N}\frac{dx}{|y|[(1+|y|)^2+|z|^2]^{\frac{N}{2}}}+O\Big(\varepsilon_{12}^{1+\tau}\Big)\\
&\quad +O\Big(\frac{1}{\lambda_1^N|\eta^2-\eta^1|^N}\Big)\int_0^{\lambda_2|\eta^2-\eta^1|/10}\Big(\frac{\lambda_1}{\lambda_2}\Big)^{\frac{N}{2}}\frac{dx}{|y|\Big((1+|y|)^2+|z|^2\Big)^{\frac{N-2}{2}}}\\
&= C_{N,k}\varepsilon_{ij}\int_{\mathbb{R}^N}\frac{dx}{|y|[(1+|y|)^2+|z|^2]^{\frac{N}{2}}}+(\varepsilon_{ij}^{1+\tau}).
\end{aligned}
$$

□

Lemma 5.2. *For $(\eta,\lambda)\in D_\mu$, μ small, we have for $j\neq i$, $j,i=1,2$,*

$$
\frac{N}{N-2}\int_{\mathbb{R}^N}\frac{1}{|y|}U_{\eta^j,\lambda_j}^{\frac{2}{N-2}}\frac{\partial U_{\eta^j,\lambda_j}}{\partial\lambda_j}U_{\eta^i,\lambda_i}dx=b_1C_N\frac{\varepsilon_{12}}{\lambda_j}+O\Big(\frac{\varepsilon_{12}^{1+\tau}}{\lambda_j}\Big),
$$

where

$$
b_1=\frac{N}{2}\int_{\mathbb{R}^N}\frac{(1-|x|^2)dx}{|y|\big[(1+|y|)^2+|z|^2\big]^{\frac{N+2}{2}}}.
$$

Proof.

$$
\begin{aligned}
&\frac{N}{N-2}\int_{\mathbb{R}^N}\frac{1}{|y|}U_{\eta^j,\lambda_j}^{\frac{2}{N-2}}\frac{\partial U_{\eta^j,\lambda_j}}{\partial\lambda_j}U_{\eta^i,\lambda_i}dx\\
=\;&\frac{N}{2\lambda_j}\int_{\mathbb{R}^N}\frac{1}{|y|}U_{\eta^j,\lambda_j}^{\frac{N}{N-2}}U_{\eta^i,\lambda_i}dx-N\lambda_1\int_{\mathbb{R}^N}\frac{1}{|y|}U_{\eta^j,\lambda_j}^{\frac{N}{N-2}}\frac{(|y|+\frac{1}{\lambda_j})|y|+|z-\eta^j|^2}{(1+\lambda_j|y|)^2+\lambda_j^2|z-\eta|^2}U_{\eta^i,\lambda_i}dx\\
=\;&:I_1-NI_2.
\end{aligned}
$$

Proceeding as done in the proof of Lemma 5.1, we find that

$$
\begin{aligned}
&I_2\\
=\;&\frac{C_{N,k}}{\lambda_j}\int_{\mathbb{R}^N}\frac{(1+|y|)|y|+|z|^2}{|y|[(1+|y|)^2+|z|^2]^{\frac{N+2}{2}}}\frac{dx}{\Big(\frac{\lambda_1}{\lambda_2}(1+\frac{\lambda_2}{\lambda_1}|y|)^2+|\sqrt{\frac{\lambda_2}{\lambda_1}}z-\sqrt{\lambda_1\lambda_2}(\eta^2-\eta^1)|^2\Big)^{\frac{N-2}{2}}}\\
=\;&\frac{C_{N,k}\varepsilon_{12}}{\lambda_j}\int_{\mathbb{R}^N}\frac{(1+|y|)|y|+|z|^2}{|y|[(1+|y|)^2+|z|^2]^{\frac{N+2}{2}}}dx+O\Big(\frac{\varepsilon_{12}^{1+\tau}}{\lambda_j}\Big).
\end{aligned}
$$

I_1 has been estimated in Lemma 5.1. As a result, we complete the proof. □

Lemma 5.3. *Under the assumptions of Lemma 5.2, we have for $j=1,2$,*

$$
\int_{\mathbb{R}^N}\frac{K(x)}{|y|}U_{\eta^j,\lambda_j}^{\frac{N}{N-2}}\frac{\partial U_{\eta^j,\lambda_j}}{\partial\lambda_j}dx\;=\;\frac{(N-2)C_{N,k}}{2\lambda_j^{\gamma_j+1}}\Big[\frac{b_2^j}{k}\sum_{i=1}^k\xi_i^j+\frac{b_3^j}{h}\sum_{i=1}^h a_i^j\Big]+O\Big(\frac{1}{\lambda_j^{\gamma_j+1+\sigma}}\Big)
$$

$$+O\Big(\frac{1}{\lambda_j}|\eta^j-\bar\eta^j|^{\gamma_j+\sigma}\Big)+O\Big(\frac{1}{\lambda_j^{\gamma_j}}|\eta^j-\bar\eta^j|\Big),$$

where

$$b_2^j=\int_{\mathbb{R}^N}\frac{|y|^{\gamma_j}(1-|y|^2-|z|^2)dx}{|y|[(1+|y|)^2+|z|^2]^N},\quad b_3^j=\int_{\mathbb{R}^N}\frac{|z|^{\gamma_j}(1-|y|^2-|z|^2)dx}{|y|[(1+|y|)^2+|z|^2]^N}.$$

Proof.

$$\begin{aligned}
&\int_{\mathbb{R}^N}\frac{K(x)}{|y|}U_{\eta^j,\lambda_j}^{\frac{N}{N-2}}\frac{\partial U_{\eta^j,\lambda_j}}{\partial\lambda_j}dx\\
=\ &\int_{\{x\in\mathbb{R}^N:|x-(0,\bar\eta^j)|\le\delta\}}\Big(\sum_{i=1}^k\xi_i^j|y_i^j|^{\gamma_j}+\sum_{i=1}^h a_i^j|z_i-\bar\eta_i^j|^{\gamma_j}+O(|x-(0,\bar\eta^j)|^{\gamma_j+\sigma})\Big)\times\\
&\times\frac{1}{|y|}U_{\eta^j,\lambda_j}^{\frac{N}{N-2}}\frac{\partial U_{\eta^j,\lambda_j}}{\partial\lambda_j}dx+O\Big(\frac{1}{\lambda_j^N}\Big)\\
=\ &\frac{C_{N,k}(N-2)}{2\lambda_j^{\gamma_j+1}}\int_{\mathbb{R}^N}\Big(\sum_{i=1}^k\xi_i^j|y_i|^{\gamma_j}+\sum_{i=1}^h a_i^j|z_i+\lambda_j(\eta_i^j-\bar\eta_i^j)|^{\gamma_j}\Big)\frac{(1-|y|^2-|z|^2)dx}{|y|[(1+|y|)^2+|z|^2]^N}\\
&+O\Big(\frac{1}{\lambda_j^{\gamma_j+1+\sigma}}\Big)+O\Big(\frac{1}{\lambda_j}|\eta^j-\bar\eta^j|^{\gamma_j+\sigma}\Big)\\
=\ &\frac{C_{N,k}(N-2)}{2\lambda_j^{\gamma_j+1}}\int_{\mathbb{R}^N}\Big(\sum_{i=1}^k\xi_i^j|y_i|^{\gamma_j}+\sum_{i=1}^h a_i^j|z_i|^{\gamma_j}\Big)\frac{(1-|y|^2-|z|^2)dx}{|y|[(1+|y|)^2+|z|^2]^N}\\
&+O\Big(\frac{1}{\lambda_j^{\gamma_j+1+\sigma}}\Big)+O\Big(\frac{1}{\lambda_j}|\eta^j-\bar\eta^j|^{\gamma_j+\sigma}\Big)+O\Big(\frac{1}{\lambda_j^{\gamma_j}}|\eta^j-\bar\eta^j|\Big)\\
=\ &\frac{C_{N,k}(N-2)}{2k\lambda_j^{\gamma_j+1}}\sum_{i=1}^k\xi_i^j\int_{\mathbb{R}^N}\frac{|y|^{\gamma_j}(1-|y|^2-|z|^2)dx}{|y|[(1+|y|)^2+|z|^2]^N}\\
&+\frac{C_{N,k}(N-2)}{2h\lambda_j^{\gamma_j+1}}\sum_{i=1}^h a_i^j\int_{\mathbb{R}^N}\frac{|z|^{\gamma_j}(1-|y|^2-|z|^2)dx}{|y|[(1+|y|)^2+|z|^2]^N}\\
&+O\Big(\frac{1}{\lambda_j^{\gamma_j+1+\sigma}}\Big)+O\Big(\frac{1}{\lambda_j}|\eta^j-\bar\eta^j|^{\gamma_j+\sigma}\Big)+O\Big(\frac{1}{\lambda_j^{\gamma_j}}|\eta^j-\bar\eta^j|\Big)\\
=\ &\frac{(N-2)C_{N,k}}{2\lambda_j^{\gamma_j+1}}\Big[\frac{b_2^j}{k}\sum_{i=1}^k\xi_i^j+\frac{b_3^j}{h}\sum_{i=1}^h a_i^j\Big]\\
&+O\Big(\frac{1}{\lambda_j^{\gamma_j+1+\sigma}}\Big)+O\Big(\frac{1}{\lambda_j}|\eta^j-\bar\eta^j|^{\gamma_j+\sigma}\Big)+O\Big(\frac{1}{\lambda_j^{\gamma_j}}|\eta^j-\bar\eta^j|\Big).
\end{aligned}$$

□

Lemma 5.4. *Suppose $(\eta,\lambda)\in D_\mu$, $v\in E_{\eta,\lambda}$. If μ and ε are small, then for $i=1,2$,*

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^N}\frac{1+\varepsilon K(x)}{|y|}\Big(\sum_{j=1}^{2}U_{\eta^j,\lambda_j}\Big)^{\frac{2}{N-2}}\frac{\partial U_{\eta^i,\lambda_i}}{\partial\lambda_i}vdx\Big|\\
= \;& O\Big(\frac{\varepsilon_{12}^{1/2+\tau}}{\lambda_i}+\frac{\varepsilon}{\lambda_i}\sum_{j=1}^{2}\Big(\frac{1}{\lambda_j^{\gamma_j}}+|\eta^j-\bar\eta^j|^{\gamma_j}\Big)\Big)\|v\|.
\end{aligned}
$$

Proof. Similarly to the proof of Lemma 4.3, we have

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^N}\frac{1+\varepsilon K(x)}{|y|}\Big(\sum_{j=1}^{2}U_{\eta^j,\lambda_j}\Big)^{\frac{2}{N-2}}\frac{\partial U_{\eta^i,\lambda_i}}{\partial\lambda_i}vdx\Big|\\
=\;&\Big|\int_{\mathbb{R}^N}\frac{1+\varepsilon K(x)}{|y|}U_{\eta^i,\lambda_i}^{\frac{2}{N-2}}\frac{\partial U_{\eta^i,\lambda_i}}{\partial\lambda_i}vdx\Big|+O\Big(\frac{\varepsilon_{12}^{1/2+\tau}}{\lambda_i}\Big)\|v\|\\
=\;&(1+\varepsilon K(0,\bar\eta^i))\Big\langle\frac{\partial U_{\eta^i,\lambda_i}}{\partial\lambda_i},\,v\Big\rangle\\
&+\varepsilon\Big|\int_{\partial\mathbb{R}^N}(K(x)-K(0,\bar\eta^i))\frac{1}{|y|}U_{\eta^i,\lambda_i}^{\frac{2}{N-2}}\frac{\partial U_{\eta^i,\lambda_i}}{\partial\lambda_i}vdx\Big|+O\Big(\frac{\varepsilon_{12}^{1/2+\tau}}{\lambda_i}\Big)\|v\|\\
=\;&O\Big(\frac{\varepsilon_{12}^{1/2+\tau}}{\lambda_i}+\frac{\varepsilon}{\lambda_i}\sum_{j=1}^{2}\Big(\frac{1}{\lambda_j^{\gamma_j}}+|\eta^j-\bar\eta^j|^{\gamma_j}\Big)\Big)\|v\|.
\end{aligned}
$$

□

Lemma 5.5. *For $(\eta,\lambda)\in D_\mu$, μ small, we have for $j\neq i$, $j,i=1,2$, $l=1,\cdots,h$,*

$$
\begin{aligned}
&\frac{N}{N-2}\int_{\mathbb{R}^N}\frac{1}{|y|}U_{\eta^j,\lambda_j}^{\frac{2}{N-2}}\frac{\partial U_{\eta^j,\lambda_j}}{\partial\eta_l^j}U_{\eta^i,\lambda_i}dx\\
=\;&\frac{C_{N,k}(N-2)}{h}\varepsilon_{12}(\eta_l^j-\eta_l^i)\int_{\mathbb{R}^N}\frac{|z|^2}{|y|[(1+|y|)^2+|z|^2]^{\frac{N+2}{2}}}dx\\
&+O\Big(\lambda_j\varepsilon_{12}^{1+\tau}\Big).
\end{aligned}
$$

Proof. The proof can be completed with the same arguments as that of estimate (F16) in [4] and Lemma 4.1. □

Lemma 5.6. *For $(\eta,\lambda)\in D_\mu$, μ small, we have for $j=1,2$, $i=1,\cdots,h$,*

$$
\begin{aligned}
\int_{\mathbb{R}^N}\frac{K(x)}{|y|}U_{\eta^j,\lambda_j}^{\frac{N}{N-2}}\frac{\partial U_{\eta^j,\lambda_j}}{\partial\eta_i^j}dx \;=\;& \frac{b_4^j(\eta_i^j-\bar\eta_i^j)a_i^j}{\lambda_j^{\gamma_j-2}}+O\Big(\frac{\lambda_j^2|\eta^j-\bar\eta^j|^2}{\lambda_j^{\gamma_j-1}}\Big)\\
&+O\Big(\frac{1}{\lambda_j^{\gamma_j-1+\sigma}}+\lambda_j|\eta^j-\bar\eta^j|^{\gamma_j+\sigma}\Big)+O\Big(\frac{1}{\lambda_j^{N-1}}\Big).
\end{aligned}
$$

where
$$b_4^j=\frac{C_{N,k}(N-2)\gamma_j}{h}\int_{\mathbb{R}^N}\frac{|z|^{\gamma_j}dx}{|y|[(1+|y|)^2+|z|^2]^N}.$$

Proof.

$$\begin{aligned}
&\int_{\mathbb{R}^N}\frac{K(x)}{|y|}U_{\eta^j,\lambda_j}^{\frac{N}{N-2}}\frac{\partial U_{\eta^j,\lambda_j}}{\partial\eta_i^j}dx\\
=\;&(N-2)\int_{\mathbb{R}^N}\frac{K(x)}{|y|}U_{\eta^j,\lambda_j}^{\frac{2(N-1)}{N-2}}\frac{\lambda_j^2(z_i-\eta_i^j)dx}{[(1+\lambda_j|y|)^2+\lambda_j^2|z-\eta^j|^2]}\\
=\;&(N-2)\int_{\{x\in\mathbb{R}^N:|x-(0,\bar\eta^j|\le\delta\}}\Big(\sum_{l=1}^k\xi_l^j|y_i|^{\gamma_j}+\sum_{l=1}^h a_l^j|z_i-\bar\eta_i^j|^{\gamma_j}+O(|x-(0,\bar\eta^j)|^{\gamma_j+\sigma})\Big)\\
&\times\frac{1}{|y|}U_{\eta^j,\lambda_j}^{\frac{2(N-1)}{N-2}}\frac{\lambda_j^2(z_i-\eta_i^j)dx}{[(1+\lambda_j|y|)^2+\lambda_j^2|z-\eta^j|^2]}+O\Big(\frac{1}{\lambda_j^{N-1}}\Big)\\
=\;&(N-2)\int_{\mathbb{R}^N}\Big(\sum_{l=1}^k\xi_l^j|y_i|^{\gamma_j}+\sum_{l=1}^h a_l^j|z_i-\bar\eta_i^j|^{\gamma_j}\Big)\frac{1}{|y|}U_{\eta^j,\lambda_j}^{\frac{2(N-1)}{N-2}}\frac{\lambda_j^2(z_i-\eta_i^j)dx}{[(1+\lambda_j|y|)^2+\lambda_j^2|z-\eta^j|^2]}\\
&O\Big(\frac{1}{\lambda_j^{\gamma_j-1+\sigma}}+\lambda_j|\eta^j-\bar\eta^j|^{\gamma_j+\sigma}\Big)+O\Big(\frac{1}{\lambda_j^{N-1}}\Big)\\
=\;&\frac{C_{N,k}(N-2)}{\lambda_j^{\gamma_j-1}}\int_{\mathbb{R}^N}\Big(\sum_{l=1}^k\xi_l^j|y_i|^{\gamma_j}+\sum_{l=1}^h a_l^j|z_i+\lambda_j(\eta_i^j-\bar\eta_i^j)|^{\gamma_j}\Big)\frac{z_idx}{|y|[(1+|y|)^2+|z|^2]^N}\\
&+O\Big(\frac{1}{\lambda_j^{\gamma_j-1+\sigma}}+\lambda_j|\eta^j-\bar\eta^j|^{\gamma_j+\sigma}\Big)+O\Big(\frac{1}{\lambda_j^{N-1}}\Big)\\
=\;&\frac{C_{N,k}(N-2)}{\lambda_j^{\gamma_j-1}}\int_{\mathbb{R}^N}\sum_{l=1}^h a_l^j\Big(|z_i|^{\gamma_j}+\gamma_j\lambda_j|z_i|^{\gamma_j-2}z_i(\eta_i^j-\bar\eta_i^j)\Big)\frac{z_idx}{|y|[(1+|y|)^2+|z|^2]^N}\\
&+O\Big(\frac{\lambda_j^2|\eta^j-\bar\eta^j|^2}{\lambda_j^{\gamma_j-1}}\Big)+O\Big(\frac{1}{\lambda_j^{\gamma_j-1+\sigma}}+\lambda_j|\eta^j-\bar\eta^j|^{\gamma_j+\sigma}\Big)+O\Big(\frac{1}{\lambda_j^{N-1}}\Big)\\
=\;&\frac{C_{N,k}(N-2)\gamma_j}{h\lambda_j^{\gamma_j-2}}(\eta_i^j-\bar\eta_i^j)a_i^j\int_{\mathbb{R}^N}\frac{|z|^{\gamma_j}dx}{|y|[(1+|y|)^2+|z|^2]^N}\\
&+O\Big(\frac{\lambda_j^2|\eta^j-\bar\eta^j|^2}{\lambda_j^{\gamma_j-1}}\Big)+O\Big(\frac{1}{\lambda_j^{\gamma_j-1+\sigma}}+\lambda_j|\eta^j-\bar\eta^j|^{\gamma_j+\sigma}\Big)+O\Big(\frac{1}{\lambda_j^{N-1}}\Big).
\end{aligned}$$

□

Lemma 5.7. *Suppose $(\eta,\lambda)\in D_\mu$, $v\in E_{\eta,\lambda}$. If μ and ε are small, then*

$$\Big|\int_{\mathbb{R}^N}\frac{1+\varepsilon K(x)}{|y|}\Big(\sum_{j=1}^2 U_{\eta^j,\lambda_j}\Big)^{\frac{2}{N-2}}\frac{\partial U_{\eta^j,\lambda_j}}{\partial\eta_i^j}vdx\Big|$$

$$= O\Big(\lambda_j \varepsilon_{12}^{1/2+\tau} + \lambda_j \varepsilon \sum_{j=1}^{2}\Big(\frac{1}{\lambda_j^{\gamma_j}} + |\eta^j - \bar\eta^j|^{\gamma_j}\Big)\|v\|.$$

Proof. The proof is similar to Lemma 5.4. □

Lemma 5.8. *Suppose $(\eta, \lambda) \in D_\mu$, $v \in E_{\eta,\lambda}$. If μ and ε are small, then*

$$\Big\langle \frac{\partial U_{\eta^j,\lambda_j}}{\partial \lambda_j}, \frac{\partial U_{\eta^i,\lambda_i}}{\partial \lambda_i} \Big\rangle = \frac{A_1}{\lambda_i\lambda_j}\delta_{ij} + \frac{C}{\lambda_i\lambda_j}\varepsilon_{ij}^{1+\tau},$$

$$\Big\langle \frac{\partial U_{\eta^j,\lambda_j}}{\partial \lambda_j}, \frac{\partial U_{\eta^i,\lambda_i}}{\partial \eta_k^i} \Big\rangle = \begin{cases} 0, & i = j \\ O(\lambda_i \varepsilon_{12}^{1+\tau}), & i \neq j, \end{cases}$$

$$\Big\langle \frac{\partial U_{\eta^j,\lambda_j}}{\partial \eta_l^j}, \frac{\partial U_{\eta^i,\lambda_i}}{\partial \eta_k^i} \Big\rangle = \begin{cases} A_2\lambda_i^2\delta_{lk} + O(\lambda_i^2\varepsilon_{12}^{1+\tau}), & i = j \\ O(\lambda_i\lambda_j\varepsilon_{12}^{1+\tau}), & i \neq j, \end{cases}$$

$$\Big\langle v, \frac{\partial^2 U_{\eta^i,\lambda_i}}{\partial \lambda_i^2} \Big\rangle = O\Big(\frac{\|v\|}{\lambda_i^2}\Big),$$

$$\Big\langle v, \frac{\partial^2 U_{\eta^i,\lambda_i}}{\partial \lambda_i \partial \eta_l^i} \Big\rangle = O(\|v\|),$$

$$\Big\langle v, \frac{\partial^2 U_{\eta^i,\lambda_i}}{\partial \eta_k^i \partial \eta_l^i} \Big\rangle = O(\lambda_i^2\|v\|).$$

Proof. The proof is similar to those of [4] and Lemma 5.1. □

Lemma 5.9. *Let b_1, b_2^j and b_3^j be defined as in Lemmas 5.2 and 5.3. Then,*

$$b_1 = -\frac{[k^2 - 2 + k(h-1) + h]N\omega_h\omega_k}{2k(k+1)} \int_0^{+\infty} \frac{s^{k-2}ds}{(1+s)^{N-h}} \int_0^{+\infty} \frac{t^{h-1}dt}{(1+t^2)^{\frac{N+2}{2}}}.$$

$$b_2^j = -\frac{(2N+2k-2)\omega_h\omega_k}{(2N-h-1)(2N-h-2)} \int_0^{+\infty} \frac{s^{\gamma_j+k-2}ds}{(1+s)^{2N-h-2}} \int_0^{+\infty} \frac{t^{h-1}dt}{(1+t^2)^N}.$$

$$b_3^j = -\frac{2\gamma_j\omega_h\omega_k}{N-\gamma_j+k-2} \int_0^{+\infty} \frac{s^{k-2}ds}{(1+s)^{N-\gamma_j+k-2}} \int_0^{+\infty} \frac{t^{\gamma_j+h-1}dt}{(1+t^2)^N}.$$

Hence, $b_1 < 0$, $b_2^j < 0$ and $b_3^j < 0$.

Proof. We only prove the estimate for b_3^j since the estimates for b_1 and b_2^j are similar.

Firstly, it is easy to check that

$$\int_0^{+\infty} \frac{s^m ds}{(1+s)^{n+1}} = \frac{n-m-1}{n} \int_0^{+\infty} \frac{s^m ds}{(1+s)^n}, \quad \forall\, 0 < m < n-1, \tag{5.1}$$

$$\int_0^{+\infty} \frac{s^{m+1} ds}{(1+s)^{n+1}} = \frac{m+1}{n-m-1} \int_0^{+\infty} \frac{s^m ds}{(1+s)^n}, \quad \forall\, 0 < m < n-1, \tag{5.2}$$

$$(5.3) \qquad \int_0^{+\infty} \frac{t^{m-2}dt}{(1+t^2)^n} = \frac{2n-m-1}{2(n-1)} \int_0^{+\infty} \frac{t^{m-2}dt}{(1+t^2)^{n-1}}, \quad \forall\, 0 < m < 2n-1.$$

Changing to polar coordinates and using the change of variable $\bar{t} = \frac{t}{1+s}$, we can find that

$$\begin{aligned}
\frac{b_3^j}{\omega_h\omega_k} &= \int_0^{+\infty}\int_0^{+\infty} \frac{s^{k-1}t^{\gamma_j+h-1}(1-s^2-t^2)}{s[(1+s)^2+t^2]^N} ds dt \\
&= \int_0^{+\infty} \frac{s^{k-2}ds}{(1+s)^{2N-\gamma_j-h}} \int_0^{+\infty} \frac{t^{\gamma_j+h-1}dt}{(1+t^2)^N} \\
&\quad - \int_0^{+\infty} \frac{s^{k}ds}{(1+s)^{2N-\gamma_j-h}} \int_0^{+\infty} \frac{t^{\gamma_j+h-1}dt}{(1+t^2)^N} \\
&\quad - \int_0^{+\infty} \frac{s^{k-2}ds}{(1+s)^{2N-\gamma_j-h-2}} \int_0^{+\infty} \frac{t^{\gamma_j+h+1}dt}{(1+t^2)^N}.
\end{aligned}$$

Inserting (5.1)-(5.3) into the above equation, we see

$$\begin{aligned}
\frac{b_3^j}{\omega_h\omega_k} &= \Big[\int_0^{+\infty} \frac{s^{k-2}ds}{(1+s)^{2N-\gamma_j-h}} - \int_0^{+\infty} \frac{s^{k}ds}{(1+s)^{2N-\gamma_j-h}} \\
&\quad - \frac{\gamma_j+h}{2N-\gamma_j-h-2} \int_0^{+\infty} \frac{s^{k-2}ds}{(1+s)^{2N-\gamma_j-h-2}}\Big] \int_0^{+\infty} \frac{t^{\gamma_j+h-1}dt}{(1+t^2)^N} \\
&= -\frac{2\gamma_j}{N-\gamma_j+k-2} \int_0^{+\infty} \frac{s^{k-2}ds}{(1+s)^{N-\gamma_j+k-2}} \int_0^{+\infty} \frac{t^{\gamma_j+h-1}dt}{(1+t^2)^N} \\
&< 0.
\end{aligned}$$

□

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